The Kodaira dimension of Hilbert modular threefolds

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ABSTRACT. Following a method introduced by Thomas-Vasquez and developed by Grundman, we prove that many Hilbert modular threefolds of geometric genus 0 and 1 are of general type, and that some are of nonnegative Kodaira dimension. The new ingredient is a detailed study of the geometry and combinatorics of totally positive integral elements x of a fractional ideal I in a totally real number field K with the property that $\operatorname{tr} xy < \min I \operatorname{tr} y$ for some $y \gg 0 \in K$.

1. Introduction

This paper is about the birational geometry, and in particular the Kodaira dimension, of Hilbert modular varieties. We begin by recalling the definition of these varieties as complex orbifolds, as in [**30**, Chapter 1], [**8**, §2.2]. Let K be a totally real number field of degree d. The standard formula $\left(\begin{smallmatrix} a & b \\ c & d \end{smallmatrix}\right) z = \frac{az+b}{cz+d}$ gives an action of $\mathrm{GL}_2^+(\mathbb{R})$ on the upper half-plane $\mathcal{H}$. The product of the d embeddings $K \hookrightarrow \mathbb{R}$ gives an injection $\mathrm{PSL}_2(\mathcal{O}_K) \hookrightarrow \mathrm{GL}_2^+(\mathbb{R})^d$ whose image is discrete and acts on $\mathcal{H}^d$ properly discontinuously and with finite stabilizers; thus the quotient inherits a structure of complex orbifold from $\mathcal{H}^d$, which is the most basic Hilbert modular variety.

REMARK 1.1. It is sometimes important to consider a quotient by $\mathrm{PGL}_2^+(\mathcal{O}_K)$, the subgroup of $\mathrm{GL}_2(\mathcal{O}_K)$ consisting of matrices whose determinant is a totally positive unit, rather than PSL_2. These groups are the same when the class number and narrow class number of K are equal, but if ε is a totally positive unit that is not a square then $\left(\begin{smallmatrix} \varepsilon & 0 \\ 0 & 1 \end{smallmatrix}\right)$ represents an element of PGL_2^+ not in PSL_2. Again the quotient is a complex orbifold of dimension d.

Let A be a fractional ideal of $\mathcal{O}_K$ and consider $\mathrm{Aut}(\mathcal{O}_K \oplus A)$. This can be viewed as a group of matrices $\left(\begin{smallmatrix} a & b \\ c & d \end{smallmatrix}\right)$, where $a, d \in \mathcal{O}_K$, $b \in A^{-1}$, $c \in A$ and where the determinant belongs to $\mathcal{O}_K^\times$. We thus write $\mathrm{PSL}(\mathcal{O}_K \oplus A), \mathrm{PGL}^+(\mathcal{O}_K \oplus A)$ for the subgroups of $\mathrm{Aut}(\mathcal{O}_K \oplus A)$ for which the determinant is the square of a unit, respectively a totally positive unit, modulo the scalar matrices.

We state some results from [**30**, I.4] on these groups. If A is generated by a totally positive element $x_+ \in K$, then $\mathrm{PSL}(\mathcal{O}_K \oplus A)$ is conjugate to PSL_2 by $\left(\begin{smallmatrix} x_+ & 0 \\ 0 & 1 \end{smallmatrix}\right)$, which identifies the action of PSL_2 on $\mathcal{H}^d$ with that of $\mathrm{PSL}(\mathcal{O}_K \oplus A)$. Similarly, if

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$A = I^2$, then the two groups are conjugate by a matrix of determinant 1 whose two columns belong to I, I^{-1}. On the other hand, the groups $\mathrm{PSL}(\mathcal{O}_K \oplus A), \mathrm{PSL}(\mathcal{O}_K \oplus B)$ are not conjugate in $\mathrm{GL}_2^+(K)$ if AB^{-1} is not of the form TI^2, where T is a narrowly principal ideal. Thus the conjugacy classes of groups of this form are in bijection with the genera of K. Similarly for PGL^+.

Definition 1.2. [**2**, (2.1.4)] Let I be a nonzero ideal of $\mathcal{O}_K$ and let A be a fractional ideal. Let $\Gamma_0(I; A), \hat{\Gamma}_0(I; A)$ be the subgroups of $\mathrm{PSL}(\mathcal{O}_K \oplus A), \mathrm{PGL}^+(\mathcal{O}_K \oplus A)$ respectively consisting of matrices whose lower left entry belongs to AI. These are subgroups of finite index, so we again obtain a complex orbifold as the quotient. This parametrizes abelian d-folds with a particular kind of action of $\mathcal{O}_K$ and a distinguished $\mathcal{O}_K$-submodule of the torsion isomorphic to $\mathcal{O}_K/I$.

Definition 1.3. Let K be a totally real number field and Γ a subgroup of finite index of $\mathrm{PGL}_2^+(\mathcal{O}_K)$ as above. The quotient $\Gamma\backslash\mathcal{H}^d$ is the *Hilbert modular variety* for the group Γ.

A priori this is only a complex orbifold, but in fact it has an algebraic structure by [**30**, Theorem II.7.1].

Notation 1.4. The Hilbert modular variety for the group $\Gamma_0(I; A)$, where I is an ideal in $\mathcal{O}_K$, will be denoted $H_{K,I;A}$. When A is (1) we will generally omit it from the notation, writing $H_{K,I}$. Similarly if $I = (1)$ we may write $H_{K;A}$, and if both are (1) we simply write H_K.

These quotients are not compact; they are compactified by adjoining finitely many points called cusps. Since our main concern is with birational geometry, we will not always distinguish between the open and compactified varieties, referring to both as Hilbert modular varieties. The cusps of H_K are in bijection with $\mathrm{PSL}_2(\mathcal{O}_K)\backslash\mathbb{P}^1(K)$, or equivalently with ideal classes of $\mathcal{O}_K$. Giving a resolution of the cusps is essentially equivalent to decomposing a fundamental domain for the action of $\mathcal{O}_{K,+}^\times$, the totally positive units of $\mathcal{O}_K$, on the totally positive cone in $K \otimes \mathbb{R}$, into cones spanned by bases for ideals. In the case $[K : \mathbb{Q}] = 2$ this is treated in great detail in [**30**, Chapter 2]; in [**25**] the problem is discussed briefly for arbitrary degree in Section 1 and studied much more closely in the rest of the paper. In contrast to the case $[K : \mathbb{Q}] = 2$ where there is a canonical way to subdivide a cone in $\mathbb{R}^2$, there are many different ways to resolve the cusp singularities in dimension greater than 2, none of which is obviously the most natural. As pointed out in [**30**, p. 36], this is related to the lack of a single minimal model for varieties of dimension greater than 2; for discussion of a related situation the reader may wish to consult [**20**, Chapter 14].

In addition, there are singularities that arise from the isolated fixed points of elements of Γ on $\mathcal{H}^d$. For $[K : \mathbb{Q}]$ fixed there are only finitely many such types and they are much more tractable than the cusps.

1.1. Previous results. Recall the definitions of Kodaira dimension κ_V and general type [**20**, Definition 1-5-2, Proposition 1-5-3]:

Notation 1.5. Let V be a smooth variety. The canonical line bundle of V will be denoted K_V; since we are not concerned with specific divisors D with $\mathcal{O}(D) \cong K_V$, we write its powers as $K_V^{\otimes n}$ rather than nK_V.

Definition 1.6. Let V be an irreducible variety and V' a smooth variety birationally equivalent to V (the choice does not matter). If $|K_{V'}^{\otimes n}| = 0$ for all $n > 0$, then we define $\kappa_V = -\infty$. If $|K_{V'}^{\otimes n}| \leq 1$ for all $n > 0$ with equality for at least one n, then define $\kappa_V = 0$. Otherwise, there exist positive integers k, n and positive real numbers c, C such that $cm^k < \dim |K_{V'}^{\otimes mn}| < Cm^k$ for sufficiently large m, and we define $\kappa_V = k$. If $\kappa_V = \dim V$ then V is *of general type*.

We now introduce two well-studied families of cubic fields.

Definition 1.7. [**25**, §3.I (*), (3.2)] Let $1 \leq r \leq s-2$. Define $K_{r,s}$ to be the field $\mathbb{Q}(\alpha_{r,s})$, where $\alpha_{r,s}$ satisfies $x(x-r)(x-s)-1=0$. Let $n > 0$. Define K_n to be $\mathbb{Q}(\alpha_n)$, where α_n is a root of $x^3+(n+1)x^2+(n-2)x-1$.

These fields have easily constructed sets of units of maximal rank: the K_n are Galois and so α_n and α_n^σ are units where $\sigma \in \mathrm{Aut}(K_n)$, while $\alpha_{r,s}, \alpha_{r,s}-r \in \mathcal{O}_{K_{r,s}}^\times$. This implies in some cases that the singularities of the Hilbert modular varieties associated to them are relatively tractable: see Remark 2.5.

Knöller proves the following [**16**, Satz 1, Satz 2]:

Theorem 1.8. (1) *If $\mathcal{O}_{K_n}$ is generated by α_n (this holds if n^2-n+7 is squarefree) and has class number* 1 *and $n > 4$, then H_{K_n} is of general type.*
(2) *If $\mathcal{O}_{K_{r,s}}$ is generated by $\alpha_{r,s}$ and has class number* 1 *and $3\zeta_{K_{r,s}}(2)D_{K_{r,s}} > 4\pi^6$, and $r > 1$, then $H_{K_{r,s}}$ is of general type.*

Every H_K that Knöller proves to be of general type satisfies $p_g(H_K) > 1$. Grundman proved the following statement:

Theorem 1.9. [**10**, Theorem 3], [**11**, Theorem 3] *Let $K \in \{K_{1,7}, K_{2,5}, K_{3,5}\}$. Then H_K is of geometric genus* 0 *and positive Kodaira dimension.*

This was surprising because a Hilbert modular surface for level 1 is rational if and only if its geometric genus is 0 [**30**, Proposition VII.6.1].

1.2. Summary of results.

We now state our main result.

Theorem 6.6, 6.7. (1) *Let K be a cubic field with discriminant ≥ 473 such the geometric genus of H_K is* 0 *or* 1*. Then H_K is of general type, unless possibly $K \cong \mathbb{Q}[x]/(x^3-7x-5)$ is the cubic field with LMFDB label 3.3.697.1.*
(2) *Let K be a cubic field such that the geometric genus of H_K is* 0 *or* 1*. Let A be an ideal that is not narrowly principal (this implies that $h_K^+ = 2$). Then $H_{K;A}$ is of general type, except possibly for the field 3.3.229.1 defined by x^3-4x-1.*

We are able to prove something about three more fields:

Proposition 6.8, 6.11. (1) *Let K_1 be the cubic field* 3.3.404.1 $= \mathbb{Q}(\alpha)$*, where α is a root of x^3-x^2-5x-1. Then $\kappa_{H_K} \geq 0$.*
(2) *Let K_2 be the cubic field* 3.3.469.1 $= \mathbb{Q}(\alpha)$*, where α is a root of x^3-x^2-5x+4. Then $\kappa_{H_K} > 0$.*
(3) *Let K_3 be the cubic field* 3.3.229.1 $= \mathbb{Q}(\alpha)$*, where α is a root of x^3-4x-1, and let A represent the nonprincipal genus. Then $\kappa_{H_{K;A}} \geq 0$.*

Remark 1.10. It would seem more natural to study the $H_{K;A}$ with $p_g \leq 1$ rather than the $H_{K;A}$ for which $p_g(H_K) \leq 1$. In Appendix A we will show that when K is a cubic field the dimensions of the spaces of modular forms for all of the $\Gamma_0(I;A)$ are equal. Since the number of cusps is the same as well, the geometric genera are equal, as we will see in Remark 1.25. So these conditions are equivalent.

Remark 1.11. Our method determines two constants $c_1(K) = -2\zeta_K(-1), c_2(K)$ associated to a field such that if $c_1(K) > c_2(K)$ then H_K is of general type. Both of these grow without extreme oscillations (see Tables 2, 3), and for the cubic field 3.3.1937.1 of maximal discriminant for which $p_g(H_K) \leq 1$ we already have $c_1(K) = 28, c_2(K) = 397/30$. Therefore we believe that if $p_g(H_K) > 1$ (which already implies that $\kappa_{H_K} > 0$) then all $H_{K;A}$ are of general type, and that this can be verified for all such K by the methods of this paper. In order to prove this it would be necessary to verify that $c_1 > c_2$ for all cubic fields violating the inequality $\frac{D_K\zeta_K(2)}{hR} \geq 2^{3-2}\pi^3$ of [**9**, Corollary 10] and all choices of A. The amount of computer time required is substantial but not prohibitive, especially since it appears that this inequality fails for only 421 cubic fields. See Section 6.1 for more details.

Previous work on the Kodaira dimension of Hilbert modular varieties of dimension greater than 2 has largely been restricted to level 1, perhaps because it is easier to compute invariants under this assumption and because $\kappa_{H_{K,I;A}} \geq \kappa_{H_{K;A}}$ (in characteristic 0 this is a general statement about dominant maps). In Propositions 7.11, 7.13 we classify the $H_{K,I}$ for which the geometric genus p_g is at most 1. We would like to prove results analogous to those of [**14**], studying the Kodaira dimensions of the $H_{K,I}$ of geometric genus at most 1. We can show that many of the $H_{K,I}$ are of general type even when we cannot prove this for H_K. Here is a sample of our results. We use $\mathfrak{p}_q$ to refer to a prime of norm q; it does not matter which one we choose because of the Galois automorphism. The full result is presented in Propositions 7.11 to 7.14. See also Tables 6, 7.

Theorem 7.6. *Let $K = \mathbb{Q}(\zeta_7)^+$. The Hilbert modular variety $H_{K,I}$ is of geometric genus 0 if and only if $I \in \{(1), \mathfrak{p}_7, (2), \mathfrak{p}_{13}, \mathfrak{p}_{29}, \mathfrak{p}_{43}\}$, and 1 if and only if $I \in \{(3), \mathfrak{p}_{41}, \mathfrak{p}_7^2, 2\mathfrak{p}_7, (4), \mathfrak{p}_{71}, \mathfrak{p}_7\mathfrak{p}_{13}, \mathfrak{p}_{97}, \mathfrak{p}_{113}, \mathfrak{p}_{127}, \mathfrak{p}_{13}^2\}$.*

Proposition 7.12, 7.17. *All of these varieties are of general type, with the possible exceptions of $I \in \{(1), \mathfrak{p}_7, (2)\}$. We have $\kappa_{H_{K,(2)}} > 0$.*

Remark 1.12. We would like to study the varieties not shown to be of general type and understand their geometry explicitly as in [**19**]. However, this seems to be a very difficult problem, and we have no new results in this direction.

1.3. Method for studying the Kodaira dimension of Hilbert modular threefolds. In order to prove that Hilbert modular threefolds are of general type, we will show that powers of the canonical bundle have many sections. Here we describe these sections as Hilbert modular forms; later, in Section 2, we explain which ones belong to $H^0(nK)$, and in Sections 3–5 we will give methods for exact and asymptotic calculation of the dimensions. In Sections 6, 7 we present the results. The statements of this section and Section 2 are not new, going back to [**16**]. However, results like those stated in Section 1.2 and proved in subsequent sections have previously appeared only in special cases.

Notation 1.13. Fix an ordering of the d real embeddings $\pi_i : K \hookrightarrow \mathbb{R}$. For $x \in K$ and $1 \leq i \leq d$, let $x_i = \pi_i(x)$. For $z \in \mathcal{H}^d$, let z_i be the ith component.

Definition 1.14. [**30**, Definition I.6.1] Let k be a nonnegative even integer. A *Hilbert modular form* of (parallel) weight $2k$ for a group Γ on an open subset $U \subseteq \mathcal{H}^d$ is a holomorphic function satisfying

$$f(\gamma z) = \left(\prod_{i=1}^{d} (c_i z_i + d_i)^{2k} (\det \gamma)_i^{-k}\right) f(z)$$

for all pairs (γ, z) with $\gamma \in \Gamma$ and $z, \gamma z \in U$. We denote the space of Hilbert modular forms of weight $2k$ by $M_{2k}(U;\Gamma)$, omitting U or Γ when these are clear or unimportant.

Remark 1.15. The books [**8, 30**] are invaluable general references for Hilbert modular forms and (for the second) Hilbert modular varieties. Magma [**5**] offers built-in functions for computing dimensions of spaces of Hilbert modular forms and coefficients of their Fourier expansions, though it is limited to certain levels for fields of odd degree. The article [**2**] gives a more advanced introduction to some computational issues for Hilbert modular forms and varieties, especially in degree 2, and to the use of the more recent software [**1**] that performs a wide range of essential computations for Hilbert modular forms over quadratic fields and Hilbert modular surfaces.

Remark 1.16. An easy calculation [**30**, Lemma III.3.1] shows that if f is a Hilbert modular form of weight $2k$ then $f(z_1, \dots, z_n)(dz_1 \dots dz_n)^{\otimes k}$ is invariant under Γ. In particular, if $\Gamma\backslash U$ is nonsingular then $\mathrm{H}^0(\Gamma\backslash U, K_U^{\otimes k})$ is naturally identified with $M_{2k}(U;\Gamma)$. However, this does not imply that the space $\mathrm{H}^0(H_K, K_U^{\otimes k})$ of modular forms of weight k can be identified with $M_{2k}(\Gamma)$, because there are additional conditions at the cusps and elliptic points for a modular form to give a section of $K_U^{\otimes k}$. Rather, as in the classical case, the space of modular forms of weight $2k$ is identified with $\mathrm{H}^0(H_K, (K(\log \text{cusps}))^{\otimes k})$, and Proj of the ring of modular forms is the *Baily-Borel compactification* of H_K [**30**, Theorem II.7.1]. In particular H_K is always of log general type.

We recall some facts about the rate of growth of the dimension of M_{2k}. Although an exact formula for the dimension requires a detailed examination of the elliptic points (Definition 1.19), the asymptotic formula does not require this. The starting point is a result of Siegel:

Theorem 1.17. [**30**, Theorem IV.1.1] *Relative to the standard hyperbolic metric on $\mathcal{H}^d$, the volume of $\mathrm{PSL}_2(\mathcal{O}_K)\backslash\mathcal{H}^d$ is equal to $2\zeta_k(-1)$.*

Specializing [**22**, Theorem 11], Thomas and Vasquez give a formula for the dimension of the space of modular forms for $\mathrm{PSL}_2(\mathcal{O}_K)$ and its torsion-free finite-index subgroups. The following well-known result follows from this:

Proposition 1.18. *Let Γ be a group commensurable with $\mathrm{PSL}_2(\mathcal{O}_K)$, where K is a totally real field of degree d. Then the dimension of the space of cusp forms of weight $2k$ is asymptotically equal to $2\zeta_K(-1)(-k)^d[\mathrm{PSL}_2(\mathcal{O}_K) : \Gamma]$.*

Proof. In the case where $\Gamma = \mathrm{PSL}_2(\mathcal{O}_K)$, this follows from [**27**, (2.2)], since in their terminology $\mathrm{PSL}_2(\mathcal{O}_K)$ is always of modular type. To pass to a commensurable subgroup we use [**8**, Theorem II.3.5] to see that the leading coefficient is proportional to $[\mathrm{PSL}_2(\mathcal{O}_K) : \Gamma]$, it being clear that the terms in the formula other than $\mathrm{vol}(\Gamma\backslash\mathcal{H}^n)(2r-1)^n$ are of lower order. $\square$

We now consider the special points of Hilbert modular varieties, namely fixed points of nonidentity elements of Γ and the cusps used to compactify $\Gamma\backslash\mathcal{H}^d$. The first of these must be studied in order to obtain an exact formula for the dimension of the space of modular forms, while the second are vital for estimating the difference between this dimension and the dimension of H^0 of powers of the canonical line bundle.

Definition 1.19. [**30**, p. 15] An *elliptic point* of $H_{K,I;A}$ is the image of the fixed point of an element of $\Gamma_0(I;A)$ of finite order greater than 1.

Notation 1.20. Let $C_{K,I;A}, E_{K,I;A}$ be the sets of cusps and elliptic points of $H_{K,I;A}$ respectively.

Definition 1.21. [**30**, Definition I.6.2] Let U be an open subset of $\mathcal{H}^d \setminus E_{K,I;A}$. Let C_U be the subset of $C_{K,I;A}$ consisting of cusps such that U contains a punctured neighbourhood of some representative of C, and similarly for E_U and elliptic points. A *Hilbert cusp form* on U is a Hilbert modular form on U that extends to a holomorphic function on $U \cup \Gamma\backslash(C_U \cup E_U)$ that is 0 on all cusps of C_U.

Theorem 1.22. [**15**, Section 3.6, Lemma] *For a resolution of singularities $\tilde{H}_{K,I;A}$ of $H_{K,I;A}$, the sections of $K_{\tilde{H}_{K,I;A}}$ are precisely the Hilbert cusp forms of weight* 2.

Hirzebruch only states this for quadratic fields, but the proof does not use this assumption. This theorem does not extend to powers of $K_{\tilde{H}_{K,I;A}}$; the sections of nK are not simply the cusp forms of weight $2n$. To describe the situation, we introduce some further notation.

Notation 1.23. Let P be a cusp or elliptic point for H. Let U_P be a small punctured neighbourhood of P (a complex manifold) and let $V_P = U_P \cup \{P\}$. Let $\tilde{V}_P$ be any resolution of the singular point P of V_P (nothing depends on the choice). The natural injection $\mathrm{H}^0(\tilde{V}_P, K_{\tilde{V}_P}^{\otimes q}) \hookrightarrow \mathrm{H}^0(U_P, K_{U_P}^{\otimes q})$ will be denoted $e_{P;q}$.

Definition 1.24. [**16**, p. 6] The *qth defect* $\delta_P(q)$ of P is $\dim \operatorname{coker} e_{P;q}$.

Remark 1.25. We have $\delta_P(0) = 0$ for all P; for elliptic points this is the case $m = 0$ of [**16**, Satz 2.0], while for cusps it is a direct consequence of [**16**, Satz 2.4] (see Definition 2.1 for an explanation of the notation). For $q = 1$, Theorem 1.22 shows that $\delta_P(1) = 1$ if P is a cusp, at least if $[K : \mathbb{Q}] > 1$. In addition, if P is an elliptic point then $\delta_P(1) = 0$. This was first shown by Freitag; it is clearly explained on [**30**, p. 56].

Now let $S = C_{K,I;A} \cup E_{K,I;A}$ and let $U = H_{K,I;A} \setminus S$, which is a smooth complex manifold. For each $P \in S$, let $\tilde{V}_P$ be a resolution of $U \cup P$, and let $\tilde{H}_{K,I;A}$ be a simultaneous resolution of all the singularities, obtained by identifying the $\tilde{V}_P$ along the open subsets identified with U. We then have $\mathrm{H}^0(\tilde{H}_{K,I;A}, K^{\otimes q}) = \bigcap_{P\in S} \operatorname{im} e_{P;q}$. It follows that $\mathrm{h}^0(\tilde{H}_{K,I;A}, K^{\otimes q}) \geq \dim M_{2q} - \sum_{P\in S} \delta_P(q)$, since $M_{2q} = \mathrm{h}^0(H_{K,I;A} \setminus S, K^{\otimes q})$ (in contrast to the case $K = \mathbb{Q}$, where we need an additional condition to ensure that modular forms are holomorphic at the cusps; see [**30**, Section I.6]). We thus obtain a lower bound for $\mathrm{h}^0(K^{\otimes q})$, namely

$$\mathrm{h}^0(K^{\otimes q}) \geq \dim M_{2q} - \sum_{P\in S} \delta_P(q). \tag{1.1}$$

If we can prove that this lower bound is positive (resp. greater than 1) for some q, then we have shown that the Kodaira dimension of the Hilbert modular variety is nonnegative (resp. positive). If we can estimate the lower bound as being at least cq^d for some $c > 0$, the variety is of general type. Grundman did this and essentially proved in [**10**, Theorem 3], [**11**, Theorem 3] that the component of the Hilbert modular threefolds corresponding to the principal genus for 3.3.697.1 is of positive Kodaira dimension, while for 3.3.761.1, 3.3.985.1 it is of general type.

REMARK 1.26. Grundman states only that the plurigenera are not equal to 0 for the principal genus, concluding that the varieties are not rational. However, the statement that H_K is of general type for the second and third fields follows from her results. We will explain this in Example 6.2 and Remark 6.3.

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2. Defects

Thomas and Vasquez give a formula [**27**, Theorem 3.10] for the dimension of M_{2q} for certain special subgroups of $\mathrm{PSL}_2(K)$. A general formula for $\Gamma_0(I)$ does not seem to appear in one single place in the literature, but can be extracted from [**31**, 39.10] and [**2**, 5.1]. The calculation is implemented in a Magma script [**18**] (I thank John Voight for many clarifying comments). This leaves the problem of computing the defects.

Thomas [**28**, Section 1], referring to [**16**], gives a method for evaluating the defect in terms of a count of elements with multiples of bounded trace. We give the details in Definition 2.1.

DEFINITION 2.1. [**16**] Let $M \subset K$ be a free abelian subgroup of rank $d = [K : \mathbb{Q}]$ and $V \subseteq \mathcal{O}_{K,+}^\times$ a group of totally positive units preserving M. Following Knöller, we consider the cusp singularity at the image of ∞ in the quotient of $\mathcal{H}^d \cup \{\infty\}$ by the group of matrices $\{(\begin{smallmatrix} v & g \\ 0 & 1 \end{smallmatrix}) : v \in V, g \in M\}$: we refer to this as a cusp of *type* (M, V). Let $\delta_{(M,V)}(q)$ be the qth defect of a cusp of type (M, V). Let $\hat{M} = \{x \in K : \operatorname{tr} xm \in \mathbb{Z} \text{ for all } m \in M\}$ be the dual of M with respect to the trace. For $q \in \mathbb{N}$, let $\Lambda_q(M) = \{x \in \hat{M}_+ : \operatorname{tr} xm < q \text{ for some } m \in M_+\} \cup \{0\}$.

For any such M, a suitable subgroup V exists:

LEMMA 2.2. *Let $M \subset K$ be a free abelian subgroup of rank d. The subgroup of $\mathcal{O}_{K,+}^\times$ preserving M is of finite index in $\mathcal{O}_{K,+}^\times$.*

PROOF. By multiplying by a suitable positive integer we may assume that $M \subset \mathcal{O}_K$. Then $\mathcal{O}_{K,+}^\times$ acts on subgroups $N \subseteq \mathcal{O}_K$ with $\mathcal{O}_K/M \cong \mathcal{O}_K/N$. There are finitely many such subgroups, so the stabilizer of any one of them is of finite index. □

In [**16**, Satz 2.4] Knöller proves:

THEOREM 2.3. *The qth defect of a cusp of type (M, V) is equal to $\#(\Lambda_q(M)/V)$.*

REMARK 2.4. In particular, we always have $\Lambda_1 = \{0\}$ and so $\delta_{M,V}(1) = 1$. This is closely related to Theorem 1.22.

REMARK 2.5. Thomas states that it is difficult to calculate $\delta_{M,V}(q)$ in general and imposes an additional restriction that implies that every element of $\Lambda_q(\mathcal{O}_K)/V$ is represented by $x \in \hat{\mathcal{O}}_{K,+}$ with $\operatorname{tr} x < q$. Later works of Thomas-Vasquez [**25**] and Grundman [**10, 11**] follow Thomas in imposing this restriction. This holds for the $K_{r,s}$ and K_n (Definition 1.7) under the additional hypothesis that a root of the given defining polynomial generates the ring of integers [**25**, Section 3]. In these cases they construct explicit cusp resolutions: see [**25**, Section 3], [**10**, Section 6].

In this work we will describe practical methods for computing $\delta_{M,V}$ without an explicit cusp resolution. This allows us to prove results analogous to those of [**25, 11**] for general totally real cubic fields.

QUESTION 2.6. Let K be a real cubic field such that for all q, every element of $\Lambda_q(\mathcal{O}_K)/V$ is represented by an element with trace less than q. Must K belong to one of the families $K_{r,s}$ or K_n? Is there a similar classification for fields of higher degree?

REMARK 2.7. The definition of the fields $K_{r,s}$ can be extended to arbitrary degree, defining $K_{r_1,\dots,r_n}$ to be the field of degree $n+1$ defined by the polynomial $x\prod_{i=1}^n (x-r_i)-1$. For $n = 1$, we obtain the quadratic fields $\mathbb{Q}(\sqrt{n^2+4})$. However, these do not necessarily have the property of Question 2.6. For example, with $n = 16$ this field is $\mathbb{Q}(\sqrt{65})$, in which $\frac{9+\sqrt{65}}{2} \cdot \frac{9-\sqrt{65}}{2} = 4$. One easily concludes that $\frac{9+\sqrt{65}}{2}$ is an element of $\Lambda_9(\mathcal{O}_K)$ having no multiple by a totally positive unit of trace less than 9. In Section 4 we will present systematic methods to find such elements or prove that they do not exist.

We now return to describe the defects of cyclic quotient singularities, following [**30**, II.6, III.3]. Van der Geer only treats the case of dimension 2, but his methods apply without change to the general situation. Thus we only state the results.

NOTATION 2.8. Let $a_1, \dots, a_n$ be positive integers relatively prime to m and less than m. Let L be the sublattice of $\mathbb{Z}^n$ consisting of vectors $(x_1, \dots, x_n)$ with $m | \sum_{i=1}^n a_i x_i$. The dual lattice, which is generated over $\mathbb{Z}^n$ by $(a_1, \dots, a_n)/m$, is denoted M. For $k = 1, \dots, m-1$ let $P_k = \frac{1}{m}(ka_j \bmod m)_{j=1}^n \in M$, and define a simplex T_k by the conditions $x_1, \dots, x_n \geq 1$ and $\sum_i P_{ki} x_i < 1$. (Note that $T_k = \emptyset$ if $\sum_i P_{ki} \geq 1$.) Let $T = \cup_i T_i$ and let $T(q)$ be T scaled up by q.

DEFINITION 2.9. Let $m \in \mathbb{Z}^+$ and let $a_1, \dots, a_n$ be positive integers less than m and relatively prime to m. Let the cyclic group C_m of order m act on $\mathbb{A}^n$ such that a generator acts by $(x_1, \dots, x_n) \to (\mu_m^{a_1} x_1, \dots, \mu_m^{a_n} x_n)$. A singularity locally isomorphic to that of $\mathbb{A}^n/C_m$ at the origin is called a *cyclic quotient singularity of type* $(a_1, \dots, a_n; m)$.

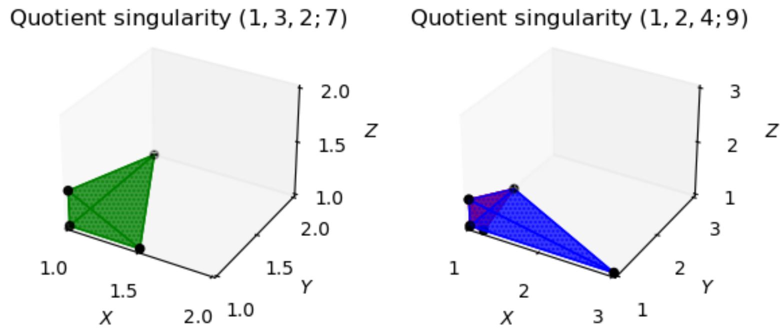

FIGURE 1. Simplices that describe the defects of $(1, 3, 2; 7)$ and $(1, 2, 4; 9)$-singularities. All simplices are defined by the condition that all coordinates are at least 1 and by one additional inequality. On the left, the choice $k = 1$ gives a single simplex defined by the inequality $x + 3y + 2z \leq 7$. On the right, the blue simplex is defined by $x + 2y + 4z \leq 9$ and the red simplex (contained in it) by $5x + y + 2z \leq 9$. Note that the scales are different.

Such a singularity is isolated by [**21**, Corollary 2.2].

THEOREM 2.10. [**30**, p. 56] *The qth defect of a cyclic quotient singularity of type* $(a_1, \ldots, a_n; m)$ *is the number of L-points of* $T(q)$.

COROLLARY 2.11. *Asymptotically the qth defect is equal to* $\frac{\operatorname{vol} T}{m} q^d + O(q^{d-1})$. □

EXAMPLE 2.12. We consider the elliptic points of order $7, 9$ that arise for $\mathbb{Q}(\zeta_7)^+$ and $\mathbb{Q}(\zeta_9)^+$. For the former, we have one point of each of the types $(1, \pm 3, \pm 2; 7)$ by [**26**, Proposition 2.10 (ii)]. The $(1, 4, 2; 7)$ point has all defects 0, while the other three are isomorphic. Indeed, given a $(1, 3, 2; 7)$-singularity, we may replace the generator g of C_7 by g^4 to see that it is also a $(4, 5, 1; 7)$-singularity, or by g^5 to see that it is a singularity of type $(5, 1, 3; 7)$. The order of the a_i is of no importance. We quickly compute that for each of them there is a single simplex of volume $1/36$, giving an asymptotic of $q^3/252$ for the qth defect. The first 10 defects are $0, 0, 0, 0, 0, 0, 1, 1, 2, 3$.

Similarly, for $\mathbb{Q}(\zeta_9)^+$, according to [**26**, Proposition 2.10 (iii)] the types are $(1, \pm 2, \pm 4; 9)$, each occurring once. All defects are 0 for the point of type $(1, 7, 4; 9)$ and again the other three are equivalent. For $(1, 2, 4; 9)$ and those isomorphic to it there are two simplices of volume $1/6, 1/60$, but the second is contained in the first so the volume of the union is just $1/6$. So the asymptotic is $q^3/54$; the initial defects come to $0, 0, 0, 0, 1, 2, 4, 6, 10, 14$.

We display the relevant simplices in Figure 1.

REMARK 2.13. As pointed out in [**16**, Satz 2.0], the defects of cyclic quotient singularities of order $2, 3$ on Hilbert modular threefolds are 0; this is immediate from Theorem 2.10 as well. Thus, in light of the classification of cyclic quotient singularities of Hilbert modular varieties of cubic fields [**27**, Theorem 2.12] (less accessibly, though earlier, in [**13**]), the cyclic quotient singularities make no contribution to the defect for fields other than $\mathbb{Q}(\zeta_7)^+, \mathbb{Q}(\zeta_9)^+$.

I thank the referee for pointing out that this is analogous to the situation for quadratic fields, for which the fields of smallest discriminant $\mathbb{Q}(\sqrt{5}), \mathbb{Q}(\sqrt{2}), \mathbb{Q}(\sqrt{3})$ admit special quotient singularities that need to be considered separately (these are described in [**30**, Table 1.III]). In general [**29**, (7)] gives all totally real fields for which there are elliptic points with nonvanishing defects. Excluding quadratic fields and quartic fields containing $\sqrt{3}$, the list is finite.

In the next two sections of this paper we describe methods for calculating the sets $\Lambda_q(M)/V$ and estimating their asymptotic growth that can be applied to general totally real fields. The most interesting case for us is that of $M = \mathcal{O}_K, V = \mathcal{O}_{K,+}^\times$, which is associated to the single cusp of H_K when K has class number 1, but imposing this restriction does not simplify the method, and we will consider more general singularities in Sections 6,7.

3. Basic algorithms

Here we describe the algorithms that underlie our computations. We assume that the standard invariants of algebraic number theory can be computed; in other words, given a number field K, we assume that its ring of integers, different, class group, and unit group can be calculated, and that the prime factorization of a fractional ideal can be determined. Such calculations can be performed by various symbolic computation systems, such as Magma (used in writing the paper and needed to run the accompanying scripts) and PARI/GP [**24**]. In addition to these, we need algorithms to find the elements of an ideal I of $\mathcal{O}_K$ whose real embeddings satisfy certain inequalities and to find the volume of a rational polytope.

3.1. Rational points in a polytope.

Definition 3.1. A *rational polytope* P in $\mathbb{R}^n$ is an intersection of half-spaces defined by linear inequalities $\sum_{i=1}^n a_i x_i \le c$ with $a_i, c \in \mathbb{Q}$ whose n-dimensional Lebesgue measure is positive and finite.

Remark 3.2. We will assume that the volume of a rational polytope can be computed given the facets. For fixed dimension, this can be done in polynomial time [**6**, p. 4033]; in this paper we are always in $\mathbb{R}^3$. By [**29**, Theorem 2], as extended in Theorem 6.12 all components of Hilbert modular varieties are of general type for fields of degree greater than 6, so in the application to Hilbert modular varieties we never need to go beyond $\mathbb{R}^6$ and the volume can be computed in polynomial time.

In order to find the integral points in a general (not necessarily rational) polytope, we begin by finding a rational polytope that contains it. This can be done from crude bounds on the individual coordinates of every point of the polytope. More sophisticated algorithms exist and have been implemented in software [**3**], but we use a simple method to find the integral points in a rational polytope:

Algorithm 3.3. *Given a rational polytope P, determine its integral points.*

1. *If* $\dim P = 1$ *the problem is trivial.*
2. *For each coordinate x_i, determine lower and upper bounds m_i, M_i for x_i. If for any i there is only one integer in $[m_i, M_i]$, then set x_i to that integer and reduce the dimension by* 1.
3. *Choose the i that maximizes $M_i - m_i$ and $c_i \in (m_i, M_i) \setminus \mathbb{Z}$ (in practice one chooses c_i near $(m_i + M_i)/2$). Apply the algorithm to the two polytopes $P \cap (x_i \ge \lceil c_i \rceil), P \cap (x_i \ge \lfloor c_i \rfloor)$.*

Remark 3.4. We have stated this for integral points, but the same algorithm can be used to determine the P-points of any rational lattice, since one can choose a rational change of coordinates taking such a lattice to $\mathbb{Z}^n$, and any such change takes a rational polytope to a rational polytope.

Remark 3.5. Magma provides a built-in function to enumerate the integral points in a polytope; however, it is often slow. Empirically we have found that combining Algorithm 3.3 with calls to the built-in function when the polytope is small enough gives better performance than either one does individually.

3.2. Union of polytopes. A second important problem is to compute the volume of a union of rational polytopes $\cup_{i=1}^k P_i$. We use the following algorithm.

Algorithm 3.6. *Given a finite set of polytopes $P_1, \ldots, P_k$, find $v = \mathrm{vol}(\cup_{i=1}^k P_i)$.*

(1) *Sort the P_i in order of increasing volume to obtain a list $\mathcal{P}_0 = \mathcal{P}$. For each pair (i,j) with $i < j$, determine whether $P_i \subseteq P_j$, and if so remove P_i from $\mathcal{P}$. For each pair (i,j) with $i < j$, determine whether $\mathrm{vol}(P_i \cap P_j) > 0$, and if so add it to the list $\mathcal{L}$ of pairs.*
(2) *While $\mathcal{L}$ is nonempty, repeat the following steps:*
    (a) *Choose an element (i,j).*
    (b) *Choose a face F of P_i that contains points of the interior of P_j (if there is no such face then $(i,j) \notin \mathcal{L}$). Let P_+, P_- be the intersections of P_j with the half-spaces defined by F.*
    (c) *Remove P_j from $\mathcal{P}$ and all pairs containing j from $\mathcal{L}$.*
    (d) *Determine whether P_+, P_- are contained in any element of $\mathcal{P}$.*
    (e) *For each $P \in \{P_+, P_-\}$ not contained in any element of $\mathcal{P}$, add P to $\mathcal{P}$, and add all pairs $(i, \pm)$ such that $P_i \cap P$ has positive volume to $\mathcal{L}$.*
(3) *Now that $\mathcal{L}$ is empty, return $\sum_{P_i \in \mathcal{P}} \mathrm{vol}\, P_i$.*

Remark 3.7. To determine whether $P_i \subseteq P_j$ in Step 1 and whether $P_\pm \subseteq P$ in Step 2d, it suffices to determine whether $\mathrm{vol}\, P = \mathrm{vol}(P_\pm \cap P)$. Since our polytopes are given by linear inequalities, we can intersect them simply by concatenating the lists of inequalities. Magma handles these issues without difficulty.

Theorem 3.8. *Algorithm 3.6 terminates and returns the volume of $\cup_{i=1}^k P_i$.*

Proof. To prove that the algorithm terminates, let Π be the set of polytopes obtained by intersecting an element of $\mathcal{P}_0$ with a collection of half-spaces determined by the facets of elements of $\mathcal{P}_0$. The set Π is finite and every polyhedron considered in the course of the algorithm belongs to Π. Further, every pass through the while-loop (Steps 2a–2e) either increases the number of elements of $\mathcal{P}$ (if neither P_+ nor P_- is contained in any other element of $\mathcal{P}$) or decreases the sum of volumes of pairwise overlaps (otherwise). Again, the order of $\mathcal{P}$ is bounded by $|\Pi|$ and the set of possible sums of volumes of pairwise overlaps is finite, being a subset of the set of sums of subsets of the intersections of pairs of elements of Π. Thus the algorithm must terminate.

We now show that the algorithm is correct. It is clear that a pass through the loop cannot alter $\cup_{P_i \in \mathcal{P}} P_i$, so the volume does not change, and that when $\mathcal{L}$ is empty the intersection of any two elements of $\mathcal{P}$ has volume 0. Thus by inclusion-exclusion the volume is $\sum_{P_i \in \mathcal{P}} \mathrm{vol}\, P_i$ at that point. □

REMARK 3.9. The upper bound on running time implied by an explicit version of the argument for termination given just above is horrifyingly large. However, in practice the running time seems quite reasonable: see Section 6 for some examples.

4. Trace-minimal elements and reducers

In this section we present the geometric and combinatorial results that underlie our estimates for the defect of the cusp singularities of a Hilbert modular variety, in particular a threefold. We use three basic concepts in the geometry of the ring of integers in a totally real number field, which we introduce here.

DEFINITION 4.1. Let K be a totally real number field and let $d = [K : \mathbb{Q}]$; we write $x \gg 0$ to mean that x is totally positive and we write K_+ for $\{x \in K : x \gg 0\}$. Let $U_+ = \{x \in \mathcal{O}_K^\times : x \gg 0\}$ be the group of totally positive units of K. Given $x \in K$, we say that x is *trace-minimal* if $\operatorname{tr} ux \ge \operatorname{tr} x$ for all $u \in U_+$.

Let $b \in \mathbb{R}^+$, and for $x \in K_+$, let $x_1, \dots, x_d$ be the real embeddings of x. If $0 < \max_i x_i / \min_i x_i \le b$ we say that x is *b-balanced*.

Finally, if I is a fractional ideal and $x \gg 0 \in I$ has the property that $\operatorname{tr} xy < \min(\mathbb{Q}^+ \cap I) \operatorname{tr} y$ for some trace-minimal $y \in K$, then x is a *reducer* relative to I, or an *I-reducer*. We abbreviate $\min(\mathbb{Q}^+ \cap I)$ to $\min I$. When $I = \mathcal{O}_K$ we refer simply to a *reducer*. The set of I-reducers will be denoted $\mathcal{R}_I$. We also denote $\mathcal{R}_I \cup \{\min I\}$ by $\mathcal{R}'_I$.

EXAMPLE 4.2. In $\mathbb{Q}(\sqrt{7})$, the equality $(3-\sqrt{7})(3+\sqrt{7}) = 2$ shows that $3 \pm \sqrt{7}$ are reducers. In Remark 2.7 we showed that $\frac{9\pm\sqrt{65}}{2}$ are reducers in $\mathbb{Q}(\sqrt{65})$.

REMARK 4.3. Not all ideals have reducers. In particular, the key property of the fields considered by Thomas-Vasquez and Grundman in [**25, 10, 11**] is that (1) has no reducers. In general, the larger the fundamental units of the number field, the less the units are capable of reducing its totally positive elements, and the more reducers it will have. For example, in $\mathbb{Q}(\sqrt{46})$, where the fundamental unit is $24335 + 3588\sqrt{46}$, there are 3542 reducers. Intuitively this is because a field with large fundamental units has a large fundamental domain for their action on the positive orthant, which allows for many elements of small trace but not too small norm which are divisible only by elements of larger trace. Note that if $r \in \mathbb{Q}^+$ then $\mathcal{R}_{rI} = r\mathcal{R}_I$.

4.1. Trace-minimal and balanced elements.

We now prove some basic properties of these definitions. Some of these results, in particular Lemma 4.6, Lemma 4.9, Corollary 4.10, are essentially contained in [**23**, Lemma 3], and our proofs are closely related as well.

DEFINITION 4.4. For this entire section, let us fix a set $S_\pm$ of units of $\mathcal{O}_K$ in bijection with the set of proper nonempty subsets of the set of real places of K such that the unit u_R corresponding to a subset R is totally positive and greater than 1 exactly at the places in R. In particular, for $1 \le i \le d$ let ${}_iu \in S_\pm$ be such that ${}_iu_j > 1$ if and only if $j = i$ (recall that ${}_iu_j$ refers to the jth real embedding of ${}_iu$). In addition, let $S_d = \{v_1, \dots, v_d\}$, where v_i is a unit that is totally positive and greater than d in the ith real embedding.

REMARK 4.5. The set $S_\pm$ can be found by standard lattice techniques. In the log embedding of the units, we are looking for units with given negative coordinates.

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One way to find such a unit is simply to enumerate units up to a given Euclidean norm in this embedding until the desired one is found. The argument does not depend on the choice, but the bounds will be better if the ${}_iu_i$ are as small as possible. Constructing S_d is easy.

LEMMA 4.6. *A trace-minimal element is totally positive or* 0.

PROOF. Let $x \in K$. If x is totally negative, choose i such that $|x_i|$ is maximal; then $\operatorname{tr}(v_i x) < dx_i < \operatorname{tr} x$. If x is positive in some real embeddings and negative in others, let u_N be as in Definition 4.4, where N is the set of real embeddings where x is negative. Then $\operatorname{tr} u_N x < \operatorname{tr} x$. □

NOTATION 4.7. Let $b_K = \max_{i \neq j} \frac{{}_iu_i - 1}{1 - {}_iu_j}$.

LEMMA 4.8. *Every totally positive element of K that is not b_K-balanced is reduced by one of the ${}_iu$.*

PROOF. Let $x \gg 0$ be trace-minimal. Suppose that x_i is the smallest embedding of x and x_j is the largest. Then

$$\operatorname{tr} x - \operatorname{tr}({}_iux) = \sum_{k=1}^{d} (x - {}_iux)_k \geq (x - {}_iux)_i + (x - {}_iux)_j = x_i(1 - {}_iu_i) + x_j(1 - {}_iu_j).$$

Now $b_K \geq \frac{{}_iu_i - 1}{1 - {}_iu_j}$, so if $\frac{x_j}{x_i} > b_K$ this is positive, and so x is reduced by ${}_iu$. □

We now describe a finite set S_K of units such that every totally positive element that is not trace-minimal is reduced by one of them.

LEMMA 4.9. *Let b_K be the bound of Notation 4.7. Let S_K consist of the units ${}_iu$ of Definition 4.4, together with all totally positive units all of whose real embeddings are at most db_K. If $x \gg 0 \in K$ satisfies $\operatorname{tr} ux \geq \operatorname{tr} x$ for all $u \in S_K$, then x is trace-minimal.*

PROOF. If x is not b_K-balanced, then by Lemma 4.8 it is reduced by one of the ${}_iu$, which belongs to S_K. Let u be a totally positive unit with $u_i > db_K$ for some i. If x is b_K-balanced, we then have $\operatorname{tr} ux \geq db_K x_i \geq db_K x_{\min} \geq dx_{\max} \geq \operatorname{tr} x$, where $x_{\min}, x_{\max}$ are the smallest and largest real embeddings of x, respectively. It follows that if x is b_K-balanced and not trace-minimal, then x is reduced by a unit whose real embeddings are at most db_K and which therefore belongs to S_K. □

We can find the finitely many totally positive units all of whose real embeddings are at most db_K: indeed, in the Minkowski embedding $x \to (\log |x_i|)$, these correspond to lattice points within a compact subset of the hyperplane $\sum_i r_i = 0$. Thus Algorithm 3.3 applies.

COROLLARY 4.10. *There is a finite set S'_K of totally positive units such that every element $x \in K$ (totally positive or not) that is not trace-minimal satisfies $\operatorname{tr} ux < \operatorname{tr} x$ for some $u \in S'_K$.*

PROOF. The sets S_K, $S_\pm$, and S_d are all finite. We choose S'_K to be their union. □

We extend the concept of trace-minimality to $K \otimes_{\mathbb{Q}} \mathbb{R}$. If we use a basis for $K \otimes_{\mathbb{Q}} \mathbb{R}$ given by elements of K, then all of the inequalities defining our polyhedra have coefficients in $\mathbb{Q}$, so Algorithm 3.3 applies and we can always determine whether an integral point in the enlarged rational polytope belongs to the original without the possibility of error due to round-off.

COROLLARY 4.11. *The set of trace-minimal elements of $K \otimes \mathbb{R}$ is a polyhedral cone with finitely many rational faces. Thus, for $q \in \mathbb{Q}^+$, the set of such elements with $\operatorname{tr} x \le q$ is a rational polytope.*

PROOF. Every element $u \in S'_K$ (Corollary 4.10) defines a half-space by $\operatorname{tr} ux \ge \operatorname{tr} x$, and the intersection of these is the trace-minimal cone. □

REMARK 4.12. In fact the trace-minimal cone is defined by $\operatorname{tr} ux \ge \operatorname{tr} x$ for $u \in S_K$. To see this, note that the two cones have the same intersection with the totally positive cone. However, every nonzero point of the trace-minimal cone is interior to the totally positive cone by Lemma 4.8. It follows that the two cones are equal.

REMARK 4.13. Let $V \subseteq \mathcal{O}_{K,+}^\times$ be a subgroup of finite index, and say that $x \in K_+$ is *V-trace-minimal* if $\operatorname{tr} vx \ge \operatorname{tr} x$ for all $v \in V$. The results and proofs of this section extend to the slightly more general situation where "trace-minimal" is replaced by "V-trace-minimal"; for simplicity we do not state these explicitly here.

We summarize the discussion of this section in an algorithm.

ALGORITHM 4.14. *Let $V \subseteq \mathcal{O}_{K,+}^\times$ be a subgroup of finite index, where K is a totally real number field. Determine the V-trace-minimal cone.*

(1) *For each real place R_i of K, determine a unit ${}_iv \in V$ greater than 1 at R_i and less than 1 at all other real places (see Remark 4.5. Let U_V be the set of these units. Calculate a constant $b_{K,V}$ as in Notation 4.7.*
(2) *Let S_V be the subset of V consisting of units less than or equal to $db_{K,V}$ at all real places. Again, this is a standard lattice calculation.*
(3) *Return $\cap_{u \in S_V \cup U_V} H_u$, where H_u is the half-space defined by $\operatorname{tr} xu \ge \operatorname{tr} x$.*

4.2. Reducers. We now consider the problem of determining the complete set of I-reducers in a totally real number field.

LEMMA 4.15. *Let b_K be the bound of Notation 4.7. Every I-reducer is at most $(\min I)db_K$ at all real places.*

PROOF. This follows from the same argument that we used in Lemma 4.9. □

Lemma 4.15 allows one to compute a finite subset of K that contains all the reducers via Algorithm 3.3, but it does not give a good bound, nor does it allow us to determine whether an element is in fact a reducer. Thus we analyze the situation more closely.

LEMMA 4.16. *Let M_K be the trace-minimal cone inside $K \otimes \mathbb{R}$, and let its extremal rays be $\mathbb{R}^+ v_i$ for $1 \le i \le m$, where $v_i \in K$. An element $x \in I$ is an I-reducer if and only if $\operatorname{tr} xv_i < \min I \operatorname{tr} v_i$ for some i.*

PROOF. The v_i are totally positive, so "if" is just the definition. For "only if", suppose that $\operatorname{tr} xy < \min I \operatorname{tr} y$ with y trace-minimal; by definition we have $y = \sum_i c_i v_i$ with $c_i \ge 0$. Thus $\operatorname{tr} xy = \sum_i c_i \operatorname{tr} xv_i$. If $\operatorname{tr} xv_i \ge \min I \operatorname{tr} v_i$ for all i, then

$$\min I \operatorname{tr} y > \operatorname{tr} xy = \sum_i c_i \operatorname{tr} xv_i \ge \min I \sum_c c_i \operatorname{tr} v_i = \min I \operatorname{tr} y.$$

This contradiction establishes "only if". □

Lemma 4.17. *Given $v \gg 0$, the elements $x \gg 0 \in I$ with $\operatorname{tr} xv < \min I \operatorname{tr} v$ are naturally in correspondence with the integral points of a simplex.*

Proof. The set of such x is the set of integral points of the region in $\mathbb{R}^d$ defined by $x_i \geq 0$ for all i and $\sum_{i=1}^d (x_i - \min I) v_i = 0$ that belong to the sublattice of $\mathbb{Z}^d$ defined by I. By changing coordinates we convert this sublattice to the standard one. □

Thus we may use Algorithm 3.3 to list the reducers efficiently. Again we summarize the discussion in an algorithm.

Algorithm 4.18. *Given a fractional ideal M, determine the M-reducers.*

(1) *Determine the trace-minimal cone (Algorithm 4.14).*
(2) *For each extremal ray $v\mathbb{R}^+$ of the trace-minimal cone, choose a representative $v \in K$ and determine the set $R_v = \{x \in M_+ : \operatorname{tr} xv < \min M \operatorname{tr} v\}$ as in Lemma 4.17.*
(3) *The answer is $\cup_v R_v$.*

5. Algorithms for defects

We now consider two closely related problems concerned with counting elements with multiples of bounded trace. Thus let M be a lattice in K and $V \subseteq \mathcal{O}_{K,+}^\times$ be a subgroup of finite index preserving M (by Lemma 2.2 such a subgroup exists). We recall the notation from Definition 2.1 and the fact [**16**, Satz 2.4] that the qth defect of a cusp of type (M, V) is $\delta_{M,V}(q) = \#\Lambda_q(M)/V$. Our problems are as follows:

Problem 5.1. Given M, V, q, compute $\delta_{M,V}(q)$.

Problem 5.2. Given M, V, give an asymptotic formula for $\delta_{M,V}(q)$.

For the first of these problems, most of the ideas of Section 4 are not necessary. By rescaling we may assume that $M \subseteq \mathcal{O}_K$. For simplicity, and because it is the only case needed in this paper by [**2**, Proposition 3.3.8], we assume that M is an ideal of $\mathcal{O}_K$.

Lemma 5.3. *Let $\mathfrak{d}_K$ be the different ideal of $\mathcal{O}_K$. Let $t \in \mathfrak{d}_K^{-1}$ and let $x \neq 0 \in \hat{M}$. Define I_1, I_2 by $(x) = \mathfrak{d}_K^{-1} M^{-1} I_1$, $(t) = \mathfrak{d}_K^{-1} I_2$. Then there exists $m \in M$ with $xm = t$ if and only if $I_1 | I_2$.*

Proof. Of course $m \in K$ with $xm = t$ is unique, so we need only determine whether $x^{-1}t \in M$. Since I_1, I_2 are integral and $(x^{-1}t) = MI_2I_1^{-1}$, the result follows. □

Lemma 5.4. *For all ideals M, all $q > 0$, and all V of finite index in $\mathcal{O}_{K,+}^\times$, we have $\delta_{M,V}(q) - 1 = [\mathcal{O}_{K,+}^\times : V]\left(\delta_{M,\mathcal{O}_{K,+}^\times} - 1\right)$.*

Proof. The quotient map $(\Lambda_q(M) \setminus \{0\})/V \to (\Lambda_q(M) \setminus \{0\})/\mathcal{O}_{K,+}^\times$ is $[\mathcal{O}_{K,+}^\times : V]$-to-1. □

Thus we obtain an algorithm to calculate the defects as follows:

Algorithm 5.5. *Given M, V where M is an ideal, compute the first q defects $(\delta_{M,V}(i))_{i=1}^q$.*

(1) *List all totally positive elements of* $\mathfrak{d}_K^{-1}$ *with trace less than* q*, using Algorithm 3.3. (We proved in Corollary 4.11 that these are the points of a lattice inside a rational polytope, so the algorithm is applicable.)*
(2) *Determine the sets of ideals* $\mathcal{I}_i = \{(t\mathfrak{d}_K) : t \in \mathfrak{d}_K^{-1}, \operatorname{tr} t < i\}$ *for* $1 \leq i \leq q$*.*
(3) *For each* $I \in \mathcal{I}_q$*, determine the set of divisors* D_I *of* I*, and let* $D_{I,+} = \{J \in D_I : IJ^{-1}M \text{ is narrowly principal}\}$*. For each* i *let* $\mathcal{D}_i = \cup_{I \in \mathcal{I}_i} D_{I,+}$*.*
(4) *The answer is* $([\mathcal{O}_{K,+}^\times : V]\#\mathcal{D}_i + 1)_{i=1}^q$*.*

Proposition 5.6. *Algorithm 5.5 terminates and is correct.*

Proof. Termination is immediate, since this algorithm has no loops and all of the steps are effective. For correctness, first we assume that $V = \mathcal{O}_{K,+}^\times$. Let $t \in \mathfrak{d}_{K,+}^{-1}$ with $\operatorname{tr} m < q$ and let $x \in \hat{M}_+$; define I_1, I_2 as in Lemma 5.3. If $tx^{-1} \in M$ then, from the above, $I_1 I_2^{-1} = t^{-1}xM^{-1}$ and $I_1 I_2^{-1} M$ is narrowly principal, being generated by $t^{-1}x$. The converse follows similarly. Counting narrowly principal ideals is the same as counting totally positive generators of those ideals up to totally positive units, so the result follows. The last step, giving the answer, is justified by Lemma 5.4. $\square$

Remark 5.7. We give the output in this form because computing the qth defect is not significantly harder than computing the first q defects.

We now consider the problem of determining an asymptotic.

Definition 5.8. For $r \gg 0 \in K$, let $T_{M,V,r}(q)$ be the intersection of the V-trace-minimal cone with the half-space $\operatorname{tr} rx \leq q$. Let $T_{M,V}(q) = \cup_{r \in \mathcal{R}'_M} T_{M,V,r}(q)$.

Example 5.9. We illustrate this definition for a quadratic and a cubic field. First let $K = \mathbb{Q}(\sqrt{14})$. The trace-minimal cone is bounded by rays through $4 \pm \sqrt{14}$, and the reducers are $4 \pm \sqrt{14}$. Thus the set $\mathcal{R}'_{\mathcal{O}_K}$ consists of 3 elements, the corresponding triangles being shown in the first plot of Figure 2. Next we consider the smallest cubic field that has reducers, namely $K = 3.3.148.1$, obtained by adjoining a root α of $x^3 - x^2 - 3x + 1$. The maximal order is $\mathbb{Z}[\alpha]$ and so we specify an element of K as a triple (a, b, c) representing $a + b\alpha + c\alpha^2$. The trace-minimal cone is bounded by rays through

$$(4, 26, 21), (15, -32, 14), (17, 18, 6), (53, -44, 10), (68, -2, -13), (139, 34, -38)$$

(one verifies that for each x in this list there are at least two totally positive units $u_x \neq 1$ with $\operatorname{tr} x = \operatorname{tr} x u_x$, an obvious necessary condition); we find 7 reducers, namely

$$(0, 1, 1), (1, -2, 1), (1, 2, 1), (2, -7, 3), (2, -3, 1), (4, 1, -1), (5, 0, -1).$$

Thus $T_{\mathcal{O}_K, \mathcal{O}_{K,+}^\times}(1)$ is a union of 8 polyhedra, shown in the second plot of Figure 2. The largest polyhedron, containing a factor of 256/275 of the volume of the union, corresponds to $T_{\mathcal{O}_K, \mathcal{O}_{K,+}^\times, 1}(1)$ and is shown in yellow. A Jupyter notebook in [**18**] contains an interactive version of this plot.

Lemma 5.10. $T_{M,V}(1)$ *is a finite union of rational polytopes and* $T_{M,V}(q)$ *is obtained by scaling* $T_{M,V}(1)$ *by* q*.* $\square$

Lemma 5.11. *Let* $x \in \hat{M}_+$ *and suppose that* $y \in M_+$ *is such that* $\operatorname{tr} xy < q$*. Then there is* $r \in \mathcal{R}'_M$ *such that* $\operatorname{tr} xr < q$*.*

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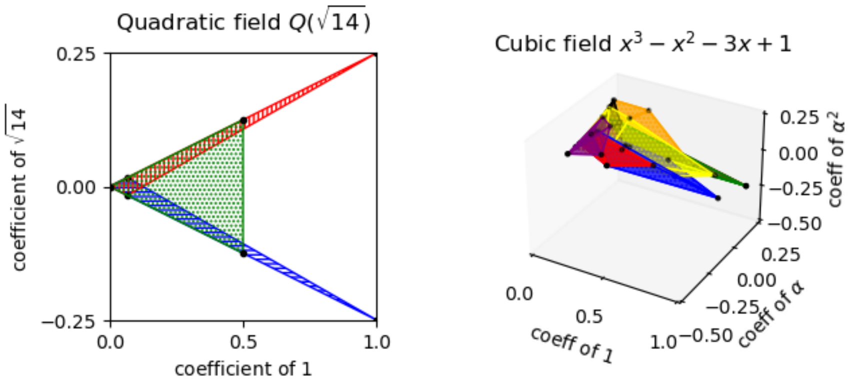

FIGURE 2. Regions describing elements of the quadratic and cubic fields of discriminant 56 and 148 respectively having an integral multiple with bounded trace. For the quadratic field we have a union of 3 triangles T_1, T_2, T_3, corresponding to the reducers $4-\sqrt{14}, 4+\sqrt{14}$ and 1, and shown in the figure in red, blue, and green respectively. The vertices of T_1 are $(0,0),(1,-1/4),(1/15,1/60)$, and those of T_3 are $(0,0),(1/2,-1/8),(1/2,1/8)$. We obtain T_2 by reflecting T_1 in the y-axis. For the cubic field, the region is a union of 8 polyhedra, each with 7 vertices.

PROOF. Suppose that $\operatorname{tr} xy < q$. If $y \in \mathcal{R}_M$ we take $r = y$. If not, then $\operatorname{tr} x < q/\min M$ and we take $r = \min M$. □

LEMMA 5.12. *There is a surjective map from the set of $\hat{M}$-points of $T_{M,V}(q)$ to $\Lambda_{q+1}(M)/V$, taking $x \in \hat{M}$ to xV. If $x \in \hat{M}$ is in the interior of $T_{M,V}(q)$, then no other element of the set has the same image as x.*

PROOF. First, this is indeed a map of the given sets: for $x \in \hat{M}$, being in $T_{M,V}(q)$ implies that there exists y with $\operatorname{tr} xy \leq q$ which is either a reducer or $\min M$, so $xV \subset \Lambda_{q+1}(M)$. For surjectivity, fix $xV \in \Lambda_{q+1}(M)/V$ and choose x to be a trace-minimal element of xV. By definition we have $\operatorname{tr} xy \leq q$ for some $y \in M$, and by Lemma 5.11 we may take $y \in \mathcal{R}'_M$. Thus $x \in T_{M,V,y}(q) \subseteq T_{M,V}(q)$.

For the injectivity statement, suppose that $x, x' \in \hat{M} \cap T_{M,V}(q)$ have the same image in $\Lambda_q(M)/V$. Since $x^{-1}x' \in V$ and x, x' are trace-minimal, we must have $\operatorname{tr} x = \operatorname{tr} x'$, which means that x, x' are on the boundary of the trace-minimal cone and hence of $T_{M,V}(q)$. □

ALGORITHM 5.13. *Given M, V, where M is a fractional ideal of K and V is a subgroup of finite index in $\mathcal{O}_{K,+}^{\times}$, compute the rational constant $c_{M,V}$ such that $\delta_{M,V}(q) \sim c_{M,V} q^d$.*

1. *Determine the trace-minimal cone by means of Algorithm 4.14.*
2. *Determine the reducers $\mathcal{R}_M$ using Algorithm 4.18.*
3. *Let $\mathcal{P}$ be the set of polyhedra given by intersecting the trace-minimal cone with the half-spaces $\operatorname{tr} rx \leq \min M \operatorname{tr} x$ for each element of $\mathcal{R}'_M$. Use Algorithm 3.6 to determine the volume E of $\cup_{P \in \mathcal{P}} P$.*

(4) *Return* $E[\mathcal{O}_{K,+}^\times : V]/N_{K/\mathbb{Q}}(\hat{M})$*, where* $N_{K/\mathbb{Q}}$ *is the norm map on fractional ideals.*

Proposition 5.14. *Algorithm 5.13 terminates and is correct.*

Proof. As before, termination is immediate from the termination of each step, since there are no loops. For correctness, by combining Lemmas 5.10, 5.12 we see that asymptotically the elements of $\Lambda_q(M)/\mathcal{O}_{K,+}^\times$ are in bijection with the points of $\hat{M}$ lying in $\cup_{P \in \mathcal{P}} P$ scaled by q. (The failure of injectivity only affects the points on polytopes of lower dimension, which are asymptotically negligible.) The correctness of the algorithm follows for $V = \mathcal{O}_{K,+}^\times$, since $\operatorname{covol} \hat{M}/\operatorname{covol} \mathcal{O}_K = N(\hat{M})$. The more general result follows from Lemma 5.4. The desired asymptotic is then a standard fact [**4**, Lemma 3.19], and since $\delta_{M,V}(q) = \#\Lambda_q(M)/V$ the result follows. □

Remark 5.15. In this algorithm we could replace the trace-minimal cone by the V-trace-minimal cone and omit the multiplication by $[\mathcal{O}_{K,+}^\times : V]$ in the last step.

Corollary 5.16. *The constant* $c_{M,V}$ *depends only on the narrow ideal class of* M*, and* $c_{M,V} = [\mathcal{O}_{K,+}^\times : V]c_{M,\mathcal{O}_{K,+}^\times}$.

Proof. Let I be narrowly principal and generated by p. There is an obvious bijection $\Lambda_q(MI)/V \leftrightarrow \Lambda_q(M)/V$ given by multiplication by p. For the second statement, if we pass to a subgroup of V of index n, the volume of the fundamental domain is multiplied by n. The result follows. □

Remark 5.17. Although $c_{pM,V} = c_{M,V}$ for $p \gg 0$, the time required to determine it using Algorithm 5.13 seems to grow rapidly with $[pM : (\min(pM))]$, and it is wise to choose p so as to minimize this quantity.

We have now solved Problems 5.1, 5.2.

Remark 5.18. Even in dimension 2, it is not true that $c_{M,V}$ is independent of M. For example, take $K = \mathbb{Q}(\sqrt{3})$ and consider the two ideals $(1), (\sqrt{3})$ that represent the narrow class group. If $M = (1)$ we find that $\delta_M(q) = 1+q(q-1)$, while for $M' = (\sqrt{3})$ it turns out that $\delta_{M'}(q) = 1 + q(q-1)/2$. By [**1**] or [**30**, Example II.5.1] the self-intersections of the curves in the cusp resolutions are respectively -4 and $-3, -2$, so this is in accordance with [**30**, Proposition III.3.6]. (Note that [**30**] refers to the obstruction to extending forms that vanish at the cusp across a resolution, not all forms, so the quantity considered by van der Geer is 1 less than our defects.) In general it appears that, as I ranges over genus representatives, the largest $c_{I,\mathcal{O}_{K,+}^\times}$ occurs for the principal genus. For examples in dimension 3, compare Tables 2 and 4.

Remark 5.19. By a standard result on rational polytopes, first proved by Ehrhart in 1962 but more easily accessible as [**4**, Theorem 3.23], the $\delta_{M,V}(q)$ for a given cusp are polynomial on residue classes. However, computations of the $\delta_{M,V}(q)$ indicate that they satisfy a simpler formula than might be expected, especially in the case $M = \mathcal{O}_K, V = \mathcal{O}_{K,+}^\times$. See Remark 6.10. We therefore suspect that there is some further structure to the $\delta(q)$ that remains to be elucidated.

6. Results at level 1

In this section we will describe the application of the algorithms presented here to Hilbert modular threefolds of level 1. To do so we need asymptotic formulas for both the dimension of the space of modular forms of weight $2k$ and the defects. For the first of these, we already have the result in Proposition 1.18. We easily derive a useful consequence:

PROPOSITION 6.1. *Suppose that* $(-1)^d \cdot 2\zeta_K(-1) > \sum_i c_i$, *where the sum ranges over the cusps of* H_K *and the* q*th defect of the* i*th cusp is asymptotic to* $c_i q^d$. *Then* H_K *is of general type.*

PROOF. We substitute the asymptotics for $\dim M_{2q}$ and $\sum_P \delta_P(q)$ into (1.1). The assumptions imply that $\mathrm{h}^0(qK)$ is not $o(q^d)$ and so H_K is of general type. □

Thus we will apply Algorithm 5.13 to determine the asymptotic rate of growth $c_2 k^3$ of the kth defect and compare it to $-2\zeta_K(-1)$. In particular, we recall that if $-2\zeta_K(-1) > c_2$ then H_K is of general type (see Remark 1.11). We begin by reexamining one of the threefolds of arithmetic genus 1 proved by Grundman [**11**] to have positive Kodaira dimension.

EXAMPLE 6.2. Let $K = K_{2,5}$ be the field obtained by adjoining a root α of $x(x-2)(x-5)-1$. As in [**11**], this field has discriminant 761, the ring of integers is generated by α, and the unit group is generated by $t, t-2, -1$, with the totally positive units generated by $t, (t-2)^2$. The class number is 1, so there is only one cusp up to equivalence, and the narrow class number is 2. We consider the variety H_K corresponding to the principal genus.

The bound b_K of Lemma 4.8 can be taken to be 76. Every totally positive element that is not trace-minimal is reduced by $(t-2)^i t^j$ for some $(i,j) \in S$, where

$$\begin{aligned} S = \{&(-2,-3), (-2,-2), (-2,-1), (-2,0), (-2,1), (-2,2), (0,-2), \\ &(0,-1), (0,1), (0,2), (0,3), (2,-1), (2,0), (2,1), (4,0)\}. \end{aligned}$$

There are no reducers for (1), so determining the trace-minimal cone is enough to calculate the defects. The rays defining this cone are spanned by

$$\begin{gathered} (23, 29, 51), \quad (350, -55, -18), \quad (682, 132, -109), \\ (-35, -375, 154), \quad (84, 139, 87), \quad (1179, 130, -165), \end{gathered}$$

where (a, b, c) abbreviates $a + b\alpha + c\alpha^2$.

Cutting the cone by the hyperplane $\operatorname{tr} x = 1$, we obtain a polyhedron whose volume is $13/4$. This matches Grundman's formula [**11**, Theorem 1], from which it follows that the qth defect of the cusp defined by the group of totally positive units is asymptotic to $13q^3/4$. Thus the defect for the group of squares of units is asymptotic to $13q^3/2$. Since $-2\zeta_K(-1) = 20/3 > 13/2$, Proposition 6.1 implies that principal component of the Hilbert modular threefold for K is of general type. On the author's laptop (a modest computational resource by the standards of the year 2025), this computation takes only 0.5 seconds.

REMARK 6.3. Similar considerations apply to $K_{3,5} = 3.3.985.1$, giving the result that the principal component of H_K is of general type. Though Grundman only states that at least one plurigenus of each of these varieties is positive, we consider the statement that they are of general type to be implicit in her work.

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Example 6.4. We now consider a more involved example: the cubic field 3.3.473.1. It is generated by a root α of $t^3 - 5t - 1$. Then $\mathcal{O}_K = \mathbb{Z}[\alpha]$, while $\mathcal{O}_K^\times = \langle -1, \alpha, \alpha+2 \rangle$ and $\mathcal{O}_{K,+}^\times = \mathcal{O}_K^{\times\,2}$. We have $h_K = h_K^+ = 1$. Every non-trace-minimal element is reduced by $\alpha^i(\alpha+2)^j$ for some $(i,j) \in S$, where

$$S = \{(-4,-2),(-2,-2),(-2,0),(-2,2),(0,-2),(0,2),(2,-2),(2,0),(2,2),(4,0)\}.$$

It turns out that there are 19 reducers, whose norms range from 3 to 15. We find that 8 of the 20 polyhedra that correspond to the elements of $\mathcal{R}'_{\mathcal{O}_K}$ are redundant, so we need only find the volume of a union of 12 polyhedra. This is small enough that we can check the result of Algorithm 3.6 by inclusion-exclusion, finding by both methods that the volume is $79/24D_K$. On the other hand, we have $-2\zeta_K(-1) = 10/3$. Since $10/3 - 79/24 > 0$, this proves that H_K is of general type. This example takes under 4 seconds, most of which is used for the volume computation, which passes through the main loop of Algorithm 3.6 (step (2)) 22 times.

We now survey the fields for which $p_g(H_K) \le 1$ [**12**, Table I], beginning with those for which we cannot prove $H_{K;I}$ to be of general type. (Although we do not have a real result for these, the information shown here will be useful in Section 7.) In the tables in this section, we use the following notation. The columns labeled h^+, r, n, t, t' refer to the narrow class number, the number of reducers, the number of passes through the main loop of Algorithm 3.6, the time for the whole calculation, and the time taken by one particular run of Algorithm 3.6. We omit the class number because it is always 1 for cubic fields K with $p_g(H_K) \le 1$ (for this reason it is also unnecessary to specify the cusp), and we omit h^+ in tables for the nonprincipal genus, since it is always 2 there. The exceptional speed of the examples of discriminant $49, 81, 169, 229, 257, 361, 697, 761, 985$, already considered by Thomas-Vasquez or Grundman [**25, 10, 11**], reflects that these are fields with no reducers and a very simple cusp resolution. The field 3.3.1489.1 is also $K_{1,8}$. Likewise, in the genus-1 case, 3.3.1369.1 is the cubic subfield of $\mathbb{Q}(\zeta_{37})$, which is K_7 (Definition 1.7) and has no reducers. On the other hand, although 3.3.1765.1, for example, can be defined by the special polynomial $p(t) = t(t-2)(t-42) - 1$, the maximal order is not generated by a root of p and the unit group is not generated by $t, t-2$, so this does not imply the existence of a particularly simple resolution.

Proposition 6.5. *For the first* 11 *cubic fields as ordered by discriminant, the zeta values and asymptotic growth of defects coming from the cusp resolution are as shown in Table 1.*

Proof. We simply apply Algorithm 3.6 to compute the growth of defects. □

Theorem 6.6. *Let K be a cubic field of discriminant at least* 473 *and narrow class number* 1 *such that the geometric genus of H_K is at most* 1*. Then the principal component of H_K and $\hat{H}_K$ is of general type, unless possibly $D(K) = 697$.*

Proof. This is computed by the method of Example 6.4. When $h^+ = 1$, there is no difference between H_K and $\hat{H}_K$. When $h^+ = 2$, the dimension of the space of cusp forms of weight $2k$ for PSL_2 is asymptotically twice that for PGL_2^+, but the stabilizer of the cusp for PSL_2 is of index 2 in that for PGL_2^+, so the defect is also asymptotically twice as large. The elliptic points do not contribute by Remark 2.13. Thus the calculation is the same for the two groups. Tables 2, 3 show the results of our computations for the $22 + 14 = 36$ real cubic fields satisfying the hypotheses

Table 1. Hilbert modular threefolds for fields of discriminant at most 469, not shown to be of general type.

K	$-2\zeta_K(-1)$	$c_{\mathcal{O}_K,\mathcal{O}_{K,+}^\times}$	h^+	r	n	t	t'
3.3.49.1	2/21	5/12	1	0	0	0.250	0.010
3.3.81.1	2/9	3/4	1	0	0	0.130	0.000
3.3.148.1	2/3	55/36	1	7	8	0.950	0.700
3.3.169.1	2/3	17/12	1	0	0	0.190	0.000
3.3.229.1	4/3	3	2	0	0	0.240	0.000
3.3.257.1	4/3	26/9	2	0	0	0.200	0.000
3.3.316.1	8/3	137/36	1	60	35	7.160	5.500
3.3.321.1	2	8/3	1	17	21	3.050	2.510
3.3.361.1	2	29/12	1	0	0	0.390	0.000
3.3.404.1	10/3	143/36	1	94	55	8.640	6.920
3.3.469.1	4	49/12	1	58	25	6.330	3.990

of the theorem. The general type result follows by noting that the second column, the constant in the asymptotic for the dimension of the space of modular forms, is greater than the third, the constant in the asymptotic for the defect. $\square$

Table 2. Hilbert modular threefolds of general type, except for discriminant 697, and geometric genus 0 (principal genus).

K	$-2\zeta_K(-1)$	$c_{\mathcal{O}_K,\mathcal{O}_{K,+}^\times}$	h^+	r	n	t	t'
3.3.473.1	10/3	79/24	1	19	22	3.750	3.020
3.3.564.1	6	1021/180	1	227	84	25.230	20.530
3.3.568.1	20/3	1141/180	1	477	60	28.130	17.700
3.3.621.1	20/3	413/72	1	219	95	34.110	28.080
3.3.697.1*	16/3	67/12	2	0	0	0.330	0.010
3.3.733.1	8	221/36	1	251	75	30.510	22.020
3.3.756.1	26/3	469/72	1	297	102	55.440	32.830
3.3.761.1	20/3	13/2	2	0	0	0.510	0.010
3.3.785.1	22/3	29/6	1	71	76	19.080	16.820
3.3.788.1	28/3	113/12	2	50	16	7.850	2.930
3.3.837.1	32/3	1381/180	1	1154	95	75.750	41.770
3.3.892.1	40/3	1091/90	2	141	54	20.220	13.610
3.3.940.1	44/3	703/72	1	1209	194	154.230	117.120
3.3.985.1	28/3	70/9	2	0	0	0.660	0.010
3.3.993.1	34/3	2377/360	1	120	68	35.030	27.330
3.3.1076.1	44/3	719/60	2	28	11	5.590	3.580
3.3.1257.1	16	101/9	2	15	15	5.880	4.540
3.3.1300.1	18	629/72	1	742	94	102.790	59.860
3.3.1345.1	46/3	779/120	1	95	143	55.150	50.250
3.3.1396.1	64/3	535/36	2	45	19	9.950	5.140
3.3.1489.1	16	134/15	2	0	0	0.920	0.000
3.3.1593.1	64/3	253/20	2	7	8	3.100	1.520

TABLE 3. Hilbert modular threefolds and geometric genus 1 (principal genus), all of general type.

K	$-2\zeta_K(-1)$	$c_{\mathcal{O}_K,\mathcal{O}_{K,+}^\times}$	h^+	r	n	t	t'
3.3.1016.1	52/3	124/9	2	842	70	85.090	29.070
3.3.1101.1	52/3	247/24	1	3198	163	278.630	125.560
3.3.1129.1	44/3	109/10	2	41	27	12.080	9.490
3.3.1229.1	56/3	37/3	2	64	43	20.580	17.330
3.3.1369.1	14	65/12	1	0	0	1.300	0.000
3.3.1373.1	68/3	1861/168	1	1351	171	173.030	109.780
3.3.1425.1	58/3	1519/180	1	481	145	97.420	72.400
3.3.1492.1	68/3	667/45	2	28	17	7.980	5.880
3.3.1573.1	76/3	2671/252	1	1088	230	234.060	179.500
3.3.1620.1	86/3	37/3	1	4708	274	623.380	300.940
3.3.1765.1	92/3	658/45	2	908	68	238.690	36.970
3.3.1825.1	68/3	401/36	2	3	4	2.810	1.230
3.3.1929.1	92/3	667/45	2	192	57	84.140	26.730
3.3.1937.1	28	397/30	2	18	26	12.890	10.270

THEOREM 6.7. *Let K be a cubic field of discriminant greater than* 229 *and narrow class number* 2 *such that the geometric genus of H_K is at most* 1*. Then the component of H_K or $\hat{H}_K$ corresponding to the nonprincipal genus is of general type.*

PROOF. Again, this is computed as above, with the same argument showing that only one of $H_K, \hat{H}_K$ need be considered. See Tables 4, 5. These tables were computed by choosing the ideal I representing the nonprincipal genus to be of minimal norm among integral ideals that are not narrowly principal. □

TABLE 4. Hilbert modular threefolds for the nonprincipal genus for fields K with $h^+ = 2$ for which $p_g(H_K) = 0$, all but the first shown to be of general type.

K	$-2\zeta_K(-1)$	$c_{\mathcal{O}_K,\mathcal{O}_{K,+}^\times}$	r	n	t	t'
3.3.229.1*	4/3	14/9	8	9	1.360	0.850
3.3.257.1	4/3	1	20	16	1.960	1.580
3.3.697.1	16/3	7/6	77	27	5.020	3.900
3.3.761.1	20/3	9/4	37	12	2.590	1.610
3.3.788.1	28/3	32/9	561	61	28.220	18.210
3.3.892.1	40/3	113/18	547	66	34.480	22.740
3.3.985.1	28/3	16/9	109	47	12.220	10.000
3.3.1076.1	44/3	127/30	281	67	25.830	20.500
3.3.1257.1	16	53/12	141	74	23.270	19.960
3.3.1396.1	64/3	40/9	1274	121	82.220	57.000
3.3.1489.1	16	4/3	182	40	17.560	12.700
3.3.1593.1	64/3	311/90	417	104	49.450	37.640

TABLE 5. Hilbert modular threefolds for the nonprincipal genus for fields K with $h^+ = 2$ for which $p_g(H_K) = 1$, all of general type.

K	$-2\zeta_K(-1)$	$c_{\mathcal{O}_K,\mathcal{O}_{K,+}^\times}$	r	n	t	t'
3.3.1016.1	52/3	73/9	3414	117	220.450	96.780
3.3.1129.1	44/3	173/36	472	39	33.160	18.250
3.3.1229.1	56/3	313/45	255	97	48.970	40.720
3.3.1492.1	68/3	35/9	850	146	90.950	67.550
3.3.1765.1	92/3	382/45	3649	100	310.810	75.830
3.3.1825.1	68/3	35/18	342	60	52.250	36.420
3.3.1929.1	92/3	53/9	1789	112	190.440	75.420
3.3.1937.1	28	21/4	134	101	73.210	64.580

We now discuss the two cubic fields for which we can prove that $\kappa_{H_K} > 0$ but not that H_K is of general type, aside from the field of discriminant 697 for which this is already known [**10**]. These correspond to the last two rows of Table 1, where $-2\zeta_K(-1)$ is less than the scaled volume of the union of polyhedra, so we cannot conclude that H_K is of general type.

PROPOSITION 6.8. (1) *Let K_1 be the cubic field* 3.3.404.1 $= \mathbb{Q}(\alpha)$, *where α is a root of $x^3 - x^2 - 5x - 1$. Then $\kappa_{H_K} \geq 0$.*
(2) *Let K_2 be the cubic field* 3.3.469.1 $= \mathbb{Q}(\alpha)$, *where α is a root of $x^3 - x^2 - 5x + 4$. Then $\kappa_{H_K} > 0$.*

PROOF. In each case we compare the dimension of the space of modular forms to the defect for small q. For K_1 we use [**27**, Theorem 3.10] to find that $\dim M_4 = 13$, while Algorithm 5.5 shows that the defect for forms of weight 4 is 12. Thus $\mathrm{h}^0(K^{\otimes 2}) \geq 1$ and $\kappa_{H_{K_1}} \geq 0$. Similarly, for K_2 the dimension and defect are 15 and 12 respectively. Thus $\mathrm{h}^0(K^{\otimes 2}) \geq 3$ and $\kappa_{H_{K_2}} > 0$. □

REMARK 6.9. For K_2, we compute that $0 \leq k \leq 7$ the dimension of M_{2k} is $1, 1, 15, 64, 172, 365, 668, 1098$ respectively. The first two defects are as always $0, 1$, but we compute the next few as $12, 60, 170, 365, 670, 1111$, and from then on they are presumably always greater than the dimension of M_{2k}, so nothing further is learned. We see that $\mathrm{h}^0(K^{\otimes 3}) \geq 4$ and $\mathrm{h}^0(K^{\otimes 4}) \geq 2$, which would also suffice to prove that $\kappa_{H_K} \geq 1$. In contrast, for K_1 the dimensions are $1, 1, 13, 53, 144, 304, 557, 915$, while the initial defects are $0, 1, 12, 59, 166, 356, 653, 1082$, and only $K^{\otimes 2}$ is seen in this way to have nonzero sections. Of course, if $K^{\otimes 2}$ has nonzero sections then so does $K^{\otimes 2n}$ for all $n \geq 0$.

REMARK 6.10. As alluded to in Remark 5.19, it appears that the defect series $\sum_{q=0}^{\infty} \delta(q) t^q$ is the rational function

$$\frac{-x^6 + 12x^5 + 38x^4 + 51x^3 + 36x^2 + 10x + 1}{(1-x)^2(1-x^2)(1-x^3)}.$$

This could be proved by a calculation like that used to prove [**10**, Theorem 1], but with much greater effort because we end up with a union of 9 convex polyhedra rather than a single one as in [**10**]. For other cubic fields, we find a similar formula, given by a polynomial of degree 6 divided by $(1-x)^2(1-x^2)(1-x^3)$. When the inverse different is replaced by some other fractional ideal containing the inverse different, it appears that the denominator changes to $(1-x)(1-x)^2(1-x^3)(1-x^d)$, where d divides the index.

We have a similar result for the smallest cubic field of narrow class number 2 and the nonprincipal genus.

PROPOSITION 6.11. *Let K_3 be the cubic field* $3.3.229.1 = \mathbb{Q}(\alpha)$, *where α is a root of x^3-4x-1, and let A represent the nonprincipal genus. Then $\kappa_{H_{K;A}} \geq 0$.*

PROOF. The argument is essentially identical to the above; we have $\dim M_4 = 6$, while the defect for the cusp relative to the full group of totally positive units is 3. By Lemma 5.4, the defect for the group of totally positive units is $2(3-1)+1=5$, and we have $\mathrm{h}^0(K^{\otimes 2}) \geq 1$. □

6.1. Fields for which the geometric genus is greater than 1. To close this section, we describe what would be required to prove that $H_{K;A}$ is of general type for all K with $p_g(H_K) > 1$. We begin by showing that the criterion of [**29**, Theorem 1] applies to all genera.

THEOREM 6.12. *Let K be a totally real field of degree $d > 2$ with class number h, regulator R, and discriminant $\mathcal{D}_K$. Suppose that K is not $\mathbb{Q}(\zeta_n)^+$ for $n \in$ $7, 9, 15, 20$, and that if $d = 4$ then 3 is not a square in K. If*

$$2^{-2d+2}\pi^{-2d}d^d\frac{d_K\zeta_K(2)}{hR} > 1$$

then all components of H_K are of general type.

PROOF. The statement is the same as that of [**29**, Theorem 1] except that we do not restrict to the principal genus and (for simplicity) do not allow the quotient by a nontrivial subgroup of $\operatorname{Aut} K$. The proof is a straightforward adaptation of Tsuyumine's to this slightly more general situation.

In our notation Tsuyumine defines $\theta(I,q) = \{\nu \in I\mathcal{D}_K^{-1} : \nu \gg 0, \mathrm{tr}(\nu\beta \leq q)$ for some $\beta \gg 0 \in I^{-1}$ (cf. Definition 2.1), according to which this would be $\Lambda_{q+1}(I^{-1}))$ and $u(I,q) = \#(\theta(I,q)/{\mathcal{O}_K^\times}^2)$. Then [**29**, Lemma 4] (again in our notation, and simplifying by ignoring the possibility of a subgroup of $\mathcal{S}_n$ acting on the coordinates) shows that $\dim S_{2k}^{(m)}(\Gamma) \geq \dim S_{2k}(\Gamma) - \sum_{\mathfrak{a}_\lambda} u(\mathfrak{a}_\lambda^2, \frac{1}{2}k+m-1)$. Here $\mathfrak{a}_\lambda$ runs over a set of ideals corresponding to the cusps and $S^{(m)}$ denotes a certain space of cusp forms [**29**, p. 271] such that if $S^{(1)} \neq 0$ and K is not one of the exceptional fields in the statement of the theorem then the Hilbert modular variety is of general type.

Let g be the number of components of the Hilbert modular variety. Asymptotically the dimension of the space of weight-$2k$ cusp forms on each component is equal by Proposition 1.18, so on each component we obtain

$$2^{-2+1}\,\pi^{-2d}\,D_K^{3/2}\,\zeta_K(2)\,(2k)^d + O(k^{d-1})$$

([**29**, p. 274]), while the inequality [**29**, Lemma 5] $u(I,q) \leq (2^{d-1}\,d^{-d}\,D_K^{1/2}\,|R|)q^d + O(q^{d-1})$ is independent of I. Now apply [**29**, Lemma 4] with $\mathfrak{a}_\lambda$ running only over ideals representing the cusps of one particular component. As in [**29**, (9)] we find that for each component

$$\dim S_{2k}^{(m)}\Gamma \geq \left(2^{-2d+1}\pi^{-2d}D_K^{3/2}\zeta_K(2) - 2^{-1}d^{-d}D_K^{1/2}hR\right)k^d + O(k^{d-1}),$$

whence as in [**29**, Theorem 1] if $2^{-2d+2}\pi^{-2d}d^d\frac{D_K\zeta_K(2)}{hR} > 1$ then the component is of general type. In this context Tsuyumine's $\hat{h}$ is equal to our h since there is no action by a nontrivial subgroup of $\mathcal{S}_n$. □

COROLLARY 6.13. *Let K be a totally real cubic field of discriminant greater than $2.77 \cdot 10^8$ and A an arbitrary genus of K. Then $H_{K;A}$ is of general type.*

PROOF. In view of Theorem 6.12, the calculation of [**9**, Theorem 2] applies equally to all genera. □

Now we would like to determine all K for which [**29**, p. 276] does not immediately show this to be the case.

HYPOTHESIS 6.14. *For all but 421 totally real cubic fields, the largest of their discriminants being 26601, the Tsuyumine-Grundman criterion $\frac{D_K\zeta_K(2)}{hR} \geq \frac{16\pi^6}{27}$ ([**29**, Theorem 1]; [**9**, Corollary 10]) is satisfied.*

In light of Grundman's observation that this holds for all K with $D_K > 2.77 \cdot 10^8$, this could be proved by a finite calculation (though one should note that the tables of cubic fields in the LMFDB [**17**] are not complete this far out). If cubic fields are ordered by discriminant, the field of discriminant 26601 is 1133rd, and we have checked that there are no further counterexamples among the first 25000 fields.

Let us assume Hypothesis 6.14. Of the 421 fields, there are 47 with $p_g(H_K) \leq 1$, and these have been studied earlier in this section. Of the remaining 374, there are 25 with class number greater than 1, so we need a lemma to describe the cusps.

LEMMA 6.15. *Let K be a totally real field and fix a genus $A \in \mathrm{Cl}^+(K)/2\,\mathrm{Cl}^+(K)$. The $h(K)$ cusps of $H_{K;A}$ are of type $(I^2A, \mathcal{O}_{K,+}^\times)$ as I runs over the ideal classes of $\mathcal{O}_K$.*

PROOF. Immediate from [**2**, Proposition 3.3.8 (a)]. □

Now we need only apply the existing code to the cusps as described above and compare their defect contribution to $-2\zeta_K(-1)$ to prove that the Hilbert modular varieties are of general type. For example, let us consider the first cubic field of class number 2, which does not satisfy the Tsuyumine-Grundman criterion.

EXAMPLE 6.16. The field $K = 3.3.1957.1$ is $\mathbb{Q}(\alpha)$, where α is a root of $x^3 - x^2 - 9x + 10$. We have $h_K = 2$, but the narrow class group is cyclic of order 4, so there are 2 genera. Let the narrow class group be represented by I_0, I_1, I_2, I_3, where I_0, I_2 are principal and I_0 is narrowly principal. We may choose the representatives to have norm $1, 2, 4, 4$ respectively (this specifies them uniquely). For the principal genus and PGL_2^+, the two cusps are of type $(I_0, \mathcal{O}_{K,+}^\times), (I_2, \mathcal{O}_{K,+}^\times)$, and for the nonprincipal genus, they are of type $(I_1, \mathcal{O}_{K,+}^\times), (I_3, \mathcal{O}_{K,+}^\times)$. If we used PSL_2 instead of PGL_2^+, the groups of units would be ${\mathcal{O}_K^\times}^2$ instead of $\mathcal{O}_{K,+}^\times$.

We calculate that $-2\zeta_K(-1) = 104/3$, and that the volume of the unions of polyhedra for cusps of type $(I_j, {\mathcal{O}_K^\times}^2)$ scaled by the covolume of $\hat{I}_j$ are respectively $34/3, 71/18, 44/15, 103/18$. Since $104/3 > 34/3 + 44/15, 71/18 + 103/18$, this shows that both $H_{K;A}$ are of general type.

HYPOTHESIS 6.17. *For all K with $p_g(H_K) > 1$ and all genus representatives A, the Hilbert modular threefold $H_{K;A}$ is of general type.*

Remark 6.18. The analogous statement is false for quadratic fields; in the honestly elliptic case of [**30**, Theorem VII.3.3] there are many counterexamples.

In [**18**] there is a script that is expected to verify that Hypothesis 6.14 implies Hypothesis 6.17 if run long enough, as well as code that will check that there are no counterexamples to Hypothesis 6.14 among cubic fields in the LMFDB.

7. Results at higher level

In this section we describe our results on the Kodaira dimension of Hilbert modular threefolds of the form $H_{K,I;A}$. As before, we can try to prove that such a surface is of general type by showing that the dimension of the space of modular forms grows faster than the defects, and if this fails we can still hope to prove that $\kappa_{H_{K,I;A}} > 0$ by finding q such that $\dim |qK| \geq 2$. If $p_g(H_K) = 1$ then we have already shown that H_K is of general type, so the same follows for its covers and so the most interesting cases are those with $p_g(H_K) = 0$.

We now begin to list the pairs K, I systematically for which the geometric genus of $H_{K,I}$ is at most 1. First note that if $J|I$ and $p_g(H_{K,J}) > 0$, then by the theory of oldforms we have $p_g(H_{K,I}) > 2p_g(H_{K,J}) \geq 2$, so such cases can be ignored. We will use the trace formula [**2**, (5.1.2)] to bound the levels for which the geometric genus is at most 1. Let us begin by stating the formula, first introducing the notation of [**2**, (5.1.3), (5.2.2)].

Notation 7.1. Let K be a totally real field of degree d and let I be an ideal of its maximal order. Let $\mathcal{S}$ be the set of orders containing one of the $\mathcal{O}_K[x]/(x^2 - tx + u)$ where u ranges over totally positive units modulo squares and t over elements of $\mathcal{O}_K$ such that $t^2 - 4u \gg 0$. For each order $S \in \mathcal{S}$, let $c_S = h(S)/2[S^\times : \mathcal{O}_K^\times]$ as in [**2**, (5.1.3)]. Let $A = (-1)^{d-1} h^+(\mathcal{O}_K)$, let $B = \frac{1}{2^{d-1}} \cdot |\zeta_K(-1)| \cdot N(I) \prod_{\mathfrak{p}|I} \left(1 + N(\mathfrak{p})^{-1}\right)$, let $C(u,t) = \frac{1}{2} \sum_{S \in \mathcal{S}} \frac{h(S)}{[S^\times : \mathcal{O}_K^\times]} m(\hat{S}, \hat{\mathcal{O}}_K; \hat{\mathcal{O}}_K^\times)$, where the m are certain embedding numbers that are products over primes dividing the level, defined in [**31**, Section 30.6], and let $D_k(u,t) = \frac{\alpha(u,t)^{k+1} - \beta(u,t)^{k+1}}{\alpha(u,t) - \beta(u,t)}$, where $\alpha(u,t), \beta(u,t)$ are the roots of $T^2 - tT + u$.

Theorem 7.2. [**2**, (5.1.2)] *The dimension of the space of Hilbert cusp forms for $\Gamma_0(I)$ of weight $2k$ is the coefficient of T^{2k} in*

$$AT^2 + BT\left(T\frac{d}{dT}\right)^n \left(\frac{T}{1-T^2}\right) + (-1)^n \sum_{(u,t)} C(u,t) \sum_{m \geq 1} N_{K/\mathbb{Q}}(D_{2m-2}(u,t)) T^{2m}.$$

Remark 7.3. We have corrected a typographical error in [**2**, (5.1.2)], where the sign before the term beginning with $\sum_{(u,t)} C(u,t)$ is omitted.

Theorem 7.4. *Let K be a totally real cubic field satisfying $p_g(H_K) = 0$ and let $\mathfrak{p}$ be a prime ideal of $\mathcal{O}_K$. If $p_g(H_{K,\mathfrak{p}}) \leq 1$ then either $\mathfrak{p}$ divides the conductor of some $S \in \mathcal{S}$ or $h^+(\mathcal{O}_K) + (N(\mathfrak{p}) + 1)\,|\zeta_K(-1)|\,/4 - 2\sum_{S \in \mathcal{S}} c_S \leq 1$.*

Remark 7.5. Of course this implies an effective upper bound for $N(\mathfrak{p})$.

Proof. This is a consequence of [**2**, (5.1.2),(5.1.3)]. Indeed, the geometric genus is the dimension of the space of weight-2 cusp forms by Theorem 1.22. We estimate the constants A, B, C, D in this case. The coefficient of T^2 in $T\left(T\frac{d}{dT}\right)^n \left(\frac{T}{1-T^2}\right)$

is always 1, and $(N(\mathfrak{p})+1)\,|\zeta_K(-1)|\,/4$ is the value of B in this context (recall that $p_g = 0$ implies $h_K = 1$ for cubic fields). The coefficient D_0 is always 1 so that factor may be ignored in computing the dimension of the space of weight-2 cusp forms, while [**31**, Lemma 30.6.17] implies that the local embedding number $m(\hat{S}, \hat{\mathcal{O}}; \hat{\mathcal{O}}^\times)$ is at most 2 when the level is an unramified prime. The result follows. □

We now present the calculation of levels at which the geometric genus is 0 or 1 in detail for the smallest cubic field.

THEOREM 7.6. *Let $K = \mathbb{Q}(\zeta_7)^+$. The Hilbert modular variety $H_{K,I}$ is of geometric genus* 0 *if and only if $I \in \{(1), \mathfrak{p}_7, (2), \mathfrak{p}_{13}, \mathfrak{p}_{29}, \mathfrak{p}_{43}\}$, and* 1 *if and only if $I \in \{(3), \mathfrak{p}_{41}, \mathfrak{p}_7^2, 2\mathfrak{p}_7, (4), \mathfrak{p}_{71}, \mathfrak{p}_7\mathfrak{p}_{13}, \mathfrak{p}_{97}, \mathfrak{p}_{113}, \mathfrak{p}_{127}, \mathfrak{p}_{13}^2\}$.*

PROOF. We apply Theorem 7.4. We have $h^+(\mathcal{O}_K) = 1$, while $\zeta_K(-1) = -1/21$ and $\sum_{S\in\mathcal{S}} c_s \le 1/2(1/2 + 2(1/3) + 6(1/7)) = 85/84$, where each term $1/2, 2(1/3), 6(1/7)$ appears in the sum if and only if $\mathfrak{p}$ splits in $K(i), K(\zeta_3), K(\zeta_7)$ respectively (we assume that $\mathfrak{p}$ is not one of the primes $\mathfrak{p}_7, (2), (3)$ dividing one of the conductors). It follows that if $N(\mathfrak{p}) > 169$ then $p_g(H_{K,\mathfrak{p}}) > 1$, and one checks cases up to there.

Next we allow the level to be the product of two primes. If the primes are distinct then the local embedding number is $\prod_{i=1}^2 1 + \left(\frac{N(\mathfrak{p}_i)}{d}\right)$, and so the contribution cannot exceed $2(85/42)$. So again if $N(\mathfrak{p}_1\mathfrak{p}_2) > 340$ we have $p_g > 1$, and it is not difficult to check up to this point. At level $\mathfrak{p}^2$ the local embedding numbers are the same as for p except for primes above $2, 3, 7$, and again it is straightforward to verify that $p_g > 1$ for primes of norm 27 or greater.

Finally, if the level is a product of three or more primes, then p_g can be less than 2 only if $p_g = 0$ for all proper divisors. However, it has already been verified that if $p_g = 0$ then I is prime, so this is not possible. □

We now study the Kodaira dimension of these covers.

DEFINITION 7.7. Let $\mathfrak{p}$ be a prime of $\mathcal{O}_K$, where K is a totally real cubic field. If $K = \mathbb{Q}(\zeta_7)^+$, then define $c_\mathfrak{p}$ to be $2, 1, 0$ according as $N(\mathfrak{p}) \equiv 1, 0, -1 \bmod 7$. If $K = \mathbb{Q}(\zeta_9)^+$, then let $c_\mathfrak{p} = 2, 1, 0$ according as $N(\mathfrak{p}) \equiv 1, 0, -1 \bmod 3$. For a general ideal I of $\mathbb{Q}(\zeta_7)^+$ or $\mathbb{Q}(\zeta_9)^+$, let $e_I = 0$ if I is divisible by the square of the ramified prime and otherwise $\frac{\prod_{\mathfrak{p}|I} c_\mathfrak{p}}{c_K}$, where $c_K = 84$ for $\mathbb{Q}(\zeta_7)^+$ and 18 for $\mathbb{Q}(\zeta_9)^+$. For all other fields define $c_\mathfrak{p} = e_I = 0$.

THEOREM 7.8. *Let K be a totally real cubic field with $h(K) = 1$, let $z = -2\zeta_K(-1)$, let A be a genus representative for K, and let v be the volume of the cusp for A as computed in Algorithm 3.6. Let $\mathfrak{p}$ be a prime ideal of $\mathcal{O}_K$ such that $(N(\mathfrak{p})+1)z > 2v + e_\mathfrak{p}$. Then $H_{K,\mathfrak{p};A}$ is of general type.*

PROOF. Just as with the modular curve $X_0(p)$, the Hilbert modular variety $H_{K,\mathfrak{p};A}$ has two cusp orbits when p is prime, corresponding to $\infty, 0$. The stabilizers are respectively the groups of upper and lower triangular matrices contained in $\Gamma_0(\mathfrak{p}; A)$; in particular, if we conjugate in $\mathrm{GL}_2(K)$ to an upper triangular subgroup of SL_2, the group of units in the stabilizer is the full group of squares of units. Since the class number is 1, it follows that the defects of both cusps are equal to the defects of the cusp for $H_{K;A}$. In view of Example 2.12, the leading coefficient in the asymptotic to the contribution of elliptic points to the defect of $H_{K,\mathfrak{p}}$ is given by $e_{\mathfrak{p}}$, since the number of quotient singularities is multiplied by $1+\left(\frac{\mathfrak{p}}{K(\alpha)}\right)$, where $K(\alpha)$ is the quadratic field corresponding to the elliptic point. On the other hand, the index of $\Gamma_0(\mathfrak{p})$ in $\mathrm{SL}_2(\mathcal{O}_K)$ is $N(\mathfrak{p})+1$, and the asymptotic dimension of the space of modular forms of weight k is multiplied by this factor. □

REMARK 7.9. The classification of cusps given here is a special case of the results of [**2**, Sections 3.1–3.3].

NOTATION 7.10. We use $\mathfrak{p}_n$ to denote a prime of norm n. If q is a rational prime that factors as $\mathfrak{P}_1\mathfrak{P}_2^2$ in $\mathcal{O}_K$, we denote $\mathfrak{P}_1, \mathfrak{P}_2$ by $\mathfrak{p}_q, \mathfrak{r}_q$ respectively.

It happens not to be necessary to distinguish the factors of primes that split completely. In the case of a Galois extension it is understood that the Galois images of all ideals mentioned have the same property. In view of Theorem A.1 in the Appendix it is unnecessary to distinguish between different genera, and so the rows referring to $3.3.229.1, 3.3.257.1$ describe both genera.

TABLE 6. Fields and primes for which $p_g(H_{K,\mathfrak{p}}) \leq 1$.

K	$\mathfrak{p} : p_g = 0$	$\mathfrak{p} : p_g = 1$
3.3.49.1	$\mathfrak{p}_7, (2), \mathfrak{p}_{13}, \mathfrak{p}_{29}, \mathfrak{p}_{43}$	$(3), \mathfrak{p}_{41}, \mathfrak{p}_{71}, \mathfrak{p}_{97}, \mathfrak{p}_{113}, \mathfrak{p}_{127}$
3.3.81.1	$\mathfrak{p}_3, \mathfrak{p}_{19}, \mathfrak{p}_{37}$	$(2), \mathfrak{p}_{17}, \mathfrak{p}_{73}$
3.3.148.1	$\mathfrak{p}_2, \mathfrak{p}_5, \mathfrak{p}_{13}$	$\mathfrak{p}_{17}, \mathfrak{p}_{25}$
3.3.169.1	$\mathfrak{p}_5, \mathfrak{p}_{13}$	(2)
3.3.229.1	$\mathfrak{p}_2, \mathfrak{p}_4, \mathfrak{p}_7$	
3.3.257.1	$\mathfrak{p}_3, \mathfrak{p}_5, \mathfrak{p}_7$	
3.3.316.1	$\mathfrak{r}_2$	$\mathfrak{p}_2$
3.3.321.1	$\mathfrak{r}_3$	$\mathfrak{p}_3, \mathfrak{p}_7$
3.3.361.1		$\mathfrak{p}_7$
3.3.404.1		$\mathfrak{p}_2$
3.3.469.1		$\mathfrak{p}_4$
3.3.568.1		$\mathfrak{r}_2$

PROPOSITION 7.11. *Table 6 gives the pairs consisting of a field and a prime $\mathfrak{p}$ for which $p_g(H_{K,\mathfrak{p}}) \leq 1$.*

PROOF. See [**18**]. Our implementation is not very efficient, but it is fast for all but the two smallest real cubic fields, so we did not feel a need to improve it. □

PROPOSITION 7.12. *The Hilbert modular varieties $H_{K,\mathfrak{p}}$ for the primes in Table 6 are of general type, except possibly as shown in Table 7.*

PROOF. Use [**18**] to check the condition of Theorem 7.8. □

PROPOSITION 7.13. *Table 8 gives the pairs consisting of a field and a nonprime ideal I for which $p_g(H_{K,I;A}) \leq 1$.*

PROOF. (Sketch.) Such an ideal can only be divisible by primes listed in Table 6 and all of its divisors other than (1) must be in one of the two tables. We verify using our implementation of [**2**, (5.1.2),(5.1.3)] that if I is in Table 8 and $\mathfrak{p}$ is in Table 6 but $I\mathfrak{p}$ is not in Table 8 then $p_g(H_{K,I}) > 1$. As in Corollary 7.11, it is unnecessary to distinguish between different genera here. Again, see [**18**]. □

TABLE 7. Fields, genera, and primes for which we cannot show that $H_{K,\mathfrak{p},I}$ is of general type. The g column is $+$ for the narrowly principal genus when $h^+ = 2$; otherwise it is left blank. There are no examples with the nonprincipal genus.

K	g	$\mathfrak{p}$
3.3.49.1		$\mathfrak{p}_7, (2)$
3.3.81.1		$\mathfrak{p}_3$
3.3.148.1		$\mathfrak{p}_2$
3.3.229.1	$+$	$\mathfrak{p}_2$
3.3.257.1	$+$	$\mathfrak{p}_3$

TABLE 8. Fields and nonprime ideals which $p_g(H_{K,I}) \leq 1$.

K	$I : p_g = 0$	$I : p_g = 1$
3.3.49.1		$\mathfrak{p}_7^2, 2\mathfrak{p}_7, (4), \mathfrak{p}_7\mathfrak{p}_{13}, \mathfrak{p}_{13}\mathfrak{p}_{13}'$
3.3.81.1	$\mathfrak{p}_3^2$	$(3), \mathfrak{p}_3\mathfrak{p}_{19}$
3.3.148.1	$\mathfrak{p}_2^2, \mathfrak{p}_2\mathfrak{p}_5$	$(2), \mathfrak{p}_2^2\mathfrak{p}_5$
3.3.169.1		$\mathfrak{p}_5\mathfrak{p}_5'$
3.3.229.1	$\mathfrak{p}_2^2, \mathfrak{p}_2^3$	
3.3.316.1	$\mathfrak{r}_2^2$	$\mathfrak{r}_2^3$
3.3.321.1	$\mathfrak{r}_3^2$	

PROPOSITION 7.14. *Let K be a cubic field for which $p_g(H_K) \leq 1$. For all proper nonprime ideals $I \subset \mathcal{O}_K$ and genus representatives A, the corresponding Hilbert modular variety is of general type, except possibly for $\mathfrak{p}_3^2$ in $\mathbb{Q}(\zeta_9)^+$, $\mathfrak{p}_2^2$ in the field of discriminant 148, and $\mathfrak{p}_2^2$ for the field of discriminant 229 (principal genus).*

PROOF. If I has a prime divisor $\mathfrak{p}$ for which $H_{K,\mathfrak{p}}$ is of general type, then $H_{K,I}$ too is of general type, so we need only consider powers of the ideals from Proposition 7.12 and $2\mathfrak{p}_7$ in $\mathbb{Q}(\zeta_7)^+$. We start with $I = \mathfrak{p}_7^2$ for $\mathbb{Q}(\zeta_7)^+$. The index of $\Gamma_0(I)$ is $N(\mathfrak{p}_7)^2 + N(\mathfrak{p}_7) = 56$, just as for the index of $\Gamma_0(p^2)$ in $\mathrm{SL}_2(\mathbb{Z})$. It is easily seen that the ray class group mod $\mathfrak{p}_7\infty_1\infty_2\infty_3$ is of order 2, and by [**2**, Corollary 3.1.18] this implies that there are 4 cusps. These are represented by $\infty, 0, 1/\pi_7, 3/\pi_7$, where $(\pi_7) = \mathfrak{p}_7$. For the first two of these the group of units is the full group of totally positive units as before, but for the other two it is the subgroup of totally positive units congruent to 1 mod $\mathfrak{p}_7$ and the defects are multiplied by the index of this subgroup, which is 3. Thus the dimension of the space of modular forms of weight $2n$ is asymptotically 56 times that for level 1, which makes it $16n^3/3$, while the defects are asymptotic to $(8 \cdot 5/12)n^3$, which is smaller. There are no elliptic points of order 7. Similarly, taking $I = (2)^2$ for $\mathbb{Q}(\zeta_7)^+$, the index is 72, the ray class group mod $2\infty_1\infty_2\infty_3$ is trivial, and so there are 3 cusps, represented by $\infty, 0, 1/2$. Again, the groups of units for the first two are ${\mathcal{O}_K^\times}^2$, but for the last we get the totally positive units congruent to 1 mod 2, a subgroup of index 7. In this case we have 8 elliptic points of order 7, of which 6 contribute to the defect. So the dimension is asymptotically $2 \cdot 72n^3/21$, while the defect is asymptotically $(9 \cdot 5/12 + 6/252)n^3$ and we have general type. We leave it to the reader to check that $H_{\mathbb{Q}(\zeta_7)^+,2\pi_7}$ is of general type, the dimension of the space of cusp forms of weight $2n$ being asymptotic to $48n^3/7$ and the defect to $71n^3/42$. (The gap is much larger in this case because there are only 4 cusps, each stabilized by the full group of units.)

For (3) in $\mathbb{Q}(\zeta_9)^+$, the index is 36 and there are 6 cusps. The stabilizers of these have index $27, 3, 3, 1, 1, 1$ in the SL_2-stabilizer, and the unit indices are respectively $1, 3, 3, 1, 1, 1$ (the 3 arises as in [**1**, Proposition 3.3.8 (a)] because the totally positive units congruent to 1 mod $\mathfrak{p}^2$ are of index 3). Since $\zeta_{\mathbb{Q}(\zeta_9)^+}(-1) = -1/9$, the dimension of the space of forms is asymptotic to $(36 \cdot 2/9)n^3$, while the defect is 10 times that for level 1 since there are no elliptic points of order 9. We compute that the defect for the cusp at level 1 is asymptotic to $3n^3/4$. Since $36 \cdot 2/9 > 10 \cdot 3/4$, the result follows.

The arguments for π_2^3 in 3.3.148.1, 3.3.229.1 are similar and we will only sketch them; likewise for π_3^2 in 3.3.257.1. The index of π_2^3 in both cases is 12 and there are 4 cusps. For 3.3.148.1, all totally positive units are 1 mod π_2^2 and so the unit indices are $1, 2, 1, 1$; thus the cusps create a defect asymptotic to $5 \cdot 55/36n^3$. On the other hand, the dimension of the space of modular forms grows like $12(-2\zeta_K(-1))n^3 = 8n^3$, which is larger.

In 3.3.229.1, not all totally positive units are 1 mod $\mathfrak{p}_2^2$ and the unit indices are all 1, so the dimension grows like $12(4/3)n^3$ and the defect like $4(3)n^3$. Again, in 3.3.257.1, the index of $\Gamma_0(\mathfrak{p}_3^2)$ is 12, not all totally positive units are 1 mod $\mathfrak{p}_3$, and there are 3 cusps with unit indices $1, 2, 1$. So the dimension of the space of modular forms and the defect grow like $12(4/3)n^3$ and $4(26/9)n^3$ respectively. □

REMARK 7.15. The argument does not apply to $\mathfrak{p}_3^2$ in $\mathbb{Q}(\zeta_9)^+$. The index is 12 and the ray class group has order 2, so again there are 4 cusps. Again the intermediate cusps $1/\pi_3, 2/\pi_3$ are stabilized by totally positive units congruent to 1 mod $\mathfrak{p}_3$, but this time that is all of them, so the defects are the same for all cusps. The dimension of the space of modular forms of weight $2n$ is 12 times that for level 1, so $8n^3/3$, while the defect is 4 times that for level 1 and is asymptotic to $3n^3$. Similarly for $\mathfrak{p}_2^2$ in 3.3.148.1; the dimension of the space of modular forms is asymptotic to $4n^3$ and the defect sum to $55n^3/12$. Again, for $\mathfrak{p}_2^2$ in 3.3.229.1, the asymptotics are $8n^3$ and $9n^3$ respectively, and we do not have a general type result.

REMARK 7.16. For reasons discussed at the end of Section 6, we certainly expect that the hypothesis that $p_g(H_K) < 1$ is unnecessary.

PROPOSITION 7.17. *In Table 9, the varieties $H_{K,I}$ have at least the indicated Kodaira dimension.*

PROOF. Table 9 also gives an n such that the dimension of the space of Hilbert cusp forms of level I and weight n is equal to resp. greater than the sum of the defects for $\kappa = 0, 1$ respectively. See [**18**] for details. □

8. Future work

It is also natural to ask about the opposite direction: namely, can it be proved that some of the Hilbert modular varieties for small fields and levels are not of general type? Unfortunately the classical methods of Hirzebruch, van de Ven, and Zagier [**30**, Chapter VII] do not apply in dimension greater than 2 because of the

TABLE 9. Fields and levels for which the Hilbert modular variety can be shown to be of nonnegative Kodaira dimension, but not of general type.

K	I	type	κ	n	$\dim M_n$	$\sum_i \delta_i(n)$
3.3.49.1	(2)		1	4	6	4
3.3.81.1	$\mathfrak{p}_3^2$		0	4	13	12
3.3.148.1	$\mathfrak{p}_2^2$		1	4	18	15
3.3.229.1	$\mathfrak{p}_2^2$	GL_2^+	1	4	33	27
3.3.229.1	$\mathfrak{p}_2^2$	SL_2	1	4	30	27
3.3.257.1	$\mathfrak{p}_3$	GL_2^+	1	4	22	18
3.3.257.1	$\mathfrak{p}_3$	SL_2	1	4	20	18

unavailability of Hirzebruch-Zagier cycles and lack of uniqueness of minimal models. To the author's knowledge there is only one such result.

THEOREM 8.1. *[Elkies-Harris, unpublished] Let $K = \mathbb{Q}(\zeta_7)^+$ or $\mathbb{Q}(\zeta_9)^+$. Then H_K is unirational.*

The proof relates H_K to a moduli space of curves of genus 2 with a point of order 7 or 9 on the Jacobian and uses an explicit construction to prove that this moduli space is rational. This approach cannot be applied to any nonabelian extension.

One might expect that there is no particular reason for the subspaces M_{2q} and S_{2q,U_C} not to be in general position inside M_{2q,U_C}. If so, then the Kodaira dimension would be $-\infty$ in all cases where the qth defect is greater than or equal to the dimension of the space of modular forms of weight $2q$. However, this expectation is violated for certain Hilbert modular surfaces, so we do not believe that it will hold for threefolds either. We give an example.

EXAMPLE 8.2. Let $K = \mathbb{Q}(\sqrt{53})$. There are no reducers and $h_K^+ = 1$. The dimension of the space of cusp forms of weight $2k$ is $7k^2/3 + O(n)$, since $\zeta_K(-1) = 7/6$. On the other hand, the area of the polygon $T_V(1)$ is $7/2$, so the kth defect is asymptotic to $7k^2/2$ (this can be verified from [**30**, Proposition III.3.6] as well; by [**30**, Example II.5.1] or [**1**] the cusp components have self-intersection $-9, -2, -2, -2, -2, -2, -2$). The 5 elliptic points of type $(3; 1, 1)$ each contribute $k^2/6$ to the defect. From [**30**, Theorem VII.3.3] the Hilbert modular surface is of Kodaira dimension 1, meaning that the nth plurigenus is asymptotically linear in n. Thus the defect conditions fail strikingly to be independent. Similar examples could be given for surfaces of general type; with $K = \mathbb{Q}(\sqrt{89})$, for example, the Hilbert modular surface is of general type and $2\zeta_K(-1) = 26/3$ while the cusp defect and sum of the elliptic defects are asymptotically $21n^2/2$ and $n^2/6$ respectively.

It therefore seems fruitful to examine the q-expansions of Hilbert modular forms for cubic fields of small discriminant not covered by Theorem 6.6 in order to obtain lower bounds on the Kodaira dimension of H_K. Such q-expansions can be computed, though at present this cannot be done quickly. In future work with Assaf, Costa, and Schiavone we will describe techniques to compute the q-expansions and their implications for the Kodaira dimension of Hilbert modular threefolds for cubic fields of small discriminant.

Appendix A. Dimensions of spaces of Hilbert modular forms over fields of odd degree (by Adam Logan and John Voight)

In this appendix, we show that for fields of odd degree, the dimension of the space of Hilbert modular forms supported on a connected component is equal across all components.

As we will be adapting results from [**2**], we compare the notation from that paper to ours. In [**2**], the totally real field, its maximal order, and its degree are F, R, n rather than $K, \mathcal{O}_K, d$, and the congruence subgroups that we have denoted $\Gamma_0(I;A), \hat{\Gamma}_0(I;A)$ are written as $\Gamma_{\mathfrak{b}} = \Gamma_0(\mathfrak{N})_{\mathfrak{b}}, \Gamma_0^1(\mathfrak{N})_{\mathfrak{b}}$, where $\mathfrak{b}$ runs over the representatives of the narrow class group indexing the connected components of the associated Hilbert modular variety.

THEOREM A.1. *Suppose that $d = [F : \mathbb{Q}]$ is odd. Then for all $k = (k_i)_{i=1}^d \in 2\mathbb{Z}_{\geq 0}$, the dimensions*

$$\dim S_k(\Gamma(I;A)), \quad \dim S_k(\hat{\Gamma}(I;A))$$

are independent of A.

PROOF. For the main part of the argument, we use a dimension formula due to Shimizu [**22**, Theorem 11] which first requires us to suppose that $k_i > 2$ for some i. For reasons of space, we do not repeat the formula here, but we claim that it depends only on the rotation factors of the elliptic points for $\Gamma_0(I;A)$ or $\hat{\Gamma}_0(I;A)$. Indeed, the leading term (a volume) depends only on the index $[\mathrm{PSL}_2(\mathcal{O}_K) : P\Gamma_0(I;A)]$ or $[\mathrm{PSL}_2(\mathcal{O}_K) : P\hat{\Gamma}_0(I;A)]$, the field K and the weight k, and the final term (cuspidal contribution) is zero as pointed out by Shimizu [**22**, (39)] using that d is odd. The key term is the middle term (elliptic points contribution): in Shimizu's notation

$$\sum_{i=1}^{s} \frac{1}{[\Gamma_{z_i} : 1]} \sum_{\gamma \in \Gamma_{z_i}, \gamma \neq 1} \prod_{i=1}^{n} \frac{{\alpha^{(i)}}^{r_i}}{1 - \alpha^{(i)}}$$

where z_i are a complete set of inequivalent elliptic points with stabilizers Γ_{z_i}, the $\alpha^{(i)}$ are the rotation factors as in [**2**, §4.3], and $r_i = k_i/2$. The number of elliptic points of each order is the same for the different genera by [**2**, Proposition 4.2.3], which gives a formula independent of A. Thus it suffices to prove that the rotation factors are the same, following [**2**, §4].

For points of order $q = 2$, there is nothing to prove, so suppose $q > 2$. From [**2**, Lemma 4.1.1], elliptic points of order $q > 2$ arise from an extension $L = K(\zeta_{2q}) \supset K$ of degree 2. Thus $K \supseteq K^+ = \mathbb{Q}(\zeta_{2q})^+$ and $[K^+ : \mathbb{Q}] = \phi(2q)/2$ divides $d = [K : \mathbb{Q}]$, which is odd. So $q = p^r$ where p is odd (and in fact, $p \equiv 3 \pmod 4$). Again since n is odd, there is a prime $\mathfrak{p}$ of R above p with odd ramification index. But p is totally ramified in $\mathbb{Q}(\zeta_{2p^r})$ with even ramification index, so $\mathfrak{p}$ must ramify in K.

Thus the oriented optimal selectivity condition [**2**, §4.3, (OOS)] fails. By [**2**, Definition 4.3.8, Theorem 4.3.11 (a)], all stabilizer orders $S = R[\gamma_i]$ are orientedly genial. In particular, the sets of rotation factors are the same across components (see the discussion following [**2**, Definition 4.3.8]) and by [**2**, Theorem 4.3.11 (c)] the multiplicities are the same as well.

We now treat the remaining cases. If $k_i = 0$ for some i, then either $k = 0$ and $\dim S_k(\Gamma_0(I;A)) = 1$ (constant functions) independent of A; otherwise $\dim S_k(\Gamma_0(I;A)) = 0$ by van der Geer [**30**, Lemma I.6.3]. Finally, we consider the case where $k_i = 2$ for all i, the most important one for us. We just showed that the polynomials P such that $P(k)$ is the dimension of the space of forms of even parallel weight k for $k > 2$ (see [**7**, §1]) are the same for all genera. The result then follows from [**7**, Satz 7.2] and the comments after it. □

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