Unconditional computation of the class groups of real quadratic fields

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Abstract. We describe an algorithm, based on the Selberg trace formula and explicit numerical computations of Maaß cusp forms, for computing the class groups and regulators of all real quadratic fields of discriminant $\Delta \leq X$ in time $O(X^{5/4+o(1)})$, without assuming any unproven conjectures. We applied the algorithm to compute up to $X = 10^{11}$ and used the output to test various implications of the Cohen–Lenstra heuristics.

1. Introduction

Let $K = \mathbb{Q}(\sqrt{\Delta})$ be the real quadratic field of discriminant Δ, Cl_Δ the ideal class group of order h_Δ (the class number), and R_Δ the regulator (natural logarithm of the fundamental unit). Several authors, including Gauß, have produced tables of these invariants, listing class numbers, and sometimes regulators and elementary divisors of the class groups, for all discriminants up to some bound; see Table 1.1 for a partial history.

Table 1.1. Notable tabulations of invariants of real quadratic fields.

Source	Bound	Invariants
Cayley 1862	$\Delta < 99$	h_Δ
Gauß 1893	$\Delta < 1000$	h_Δ
Saito and Wada 1988 [22]	$\Delta < 10^6$	Cl_Δ, R_Δ
Jacobson 1998 [15]	$\Delta < 10^9$	Cl_Δ, R_Δ
te Riele and Williams 2003 [27]	$p < 2 \cdot 10^{11}$, $p \equiv 1 \pmod 4$	h_Δ, R_Δ

Having extensive tables of class groups and regulators is useful for a variety of reasons. For example, Gauß' famous conjecture that there are infinitely many real quadratic fields with class number one was made based on his data. The Cohen–Lenstra heuristics [9] were also inspired by numerical data. Today, it is of interest to extend known tables even further to strengthen existing numerical evidence in support of these conjectures and others. In addition, tabulating class groups frequently uncovers new examples with rare and interesting properties, such as the only known example of a real quadratic field whose class group has 5-rank equal to 3 [15].

One important aspect of such tabulations is that, if the purpose is to provide numerical evidence in support of unproven conjectures, it is desirable that the class groups and regulators computed be unconditionally correct. This is especially important for cases where the conjectures being tested are conditional on the same hypothesis required for the correctness of the data, for example Littlewood's bounds on extreme values of L-functions

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dependent on the extended Riemann hypothesis. Unfortunately, the correctness of the fastest algorithms for computing class groups of real quadratic fields is also dependent on the extended Riemann hypothesis, and these conditional algorithms are significantly faster than their unconditional counterparts.

Thus, it is tempting to use the faster conditionally correct algorithms for class group tabulation because this enables the computation of much larger tables. For imaginary quadratic fields, Ramachandran, Jacobson, and Williams [16] resolved this issue by employing a batch verification algorithm, using the Eichler–Selberg trace formula for holomorphic cusp forms to verify the correctness of the entire table of class groups in a post-processing operation, allowing the tables of class groups to be extended significantly in that case.

In this paper, we follow the same approach for real quadratic fields, using the Selberg trace formula for Maaß forms as the basis for a novel algorithm for verifying a table of class groups and regulators of real quadratic fields. This, combined with a modification of the generic group structure algorithm of Buchmann and Schmidt [8] for producing the table of class groups, allowed us to extend significantly the table of known class groups to include all fields of discriminant up to 1.1×10^{11}. Most importantly, thanks to the new verification algorithm, our results are unconditionally correct for $\Delta \le 10^{11}$, requiring no assumptions of Riemann hypotheses. Following [15], our class group data was used to test several unproved conjectures, for which no discrepancies were found. In addition, some new examples of class groups with rare and unusual properties were discovered, including the real quadratic field of smallest discriminant whose class group has 3-rank ≥ 4 ($\mathrm{Cl}_\Delta \cong \mathbb{Z}/2\mathbb{Z} \times (\mathbb{Z}/3\mathbb{Z})^4$ for $\Delta = 58\,343\,207\,081$).

The main difference between the present work and the imaginary quadratic case in [16] is that the trace formula for holomorphic forms isolates a fixed weight, resulting in a finite-dimensional space. In fact, the approach in [16] used a space of dimension 0, so no modular form computations were needed in order to compute traces. By contrast, the trace formula for Maaß forms necessarily involves infinitely many forms. In practice this means that we need to truncate certain infinite sums and estimate the error, and we require explicit, rigorous, numerical computations of Maaß forms [2, 3].

In the remaining parts of the paper, we first describe our new verification method, followed by our method used to compute the class groups and regulators and numerical results.

2. Maaß forms and the Selberg trace formula

Let $\mathbb{H} = \{z = x + iy \in \mathbb{C} : y > 0\}$ denote the hyperbolic upper half-plane. This is acted on by the modular group $\Gamma = \mathrm{PSL}(2, \mathbb{Z})$ via linear fractional transformations. A Maaß cusp form for Γ is a non-constant, smooth function $f : \mathbb{H} \to \mathbb{C}$ that satisfies the following properties:

(1) $f(\gamma z) = f(z)$ for all $z \in \mathbb{H}$ and $\gamma \in \Gamma$,
(2) $f \in L^2(\Gamma\backslash\mathbb{H})$,
(3) f is an eigenfunction of the Laplace–Beltrami operator Δ on $\mathbb{H}$ given by

$$\Delta = -y^2 \left(\frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial y^2} \right).$$

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Furthermore, we define the Hecke operators T_n by

$$T_n f(z) = \frac{1}{\sqrt{|n|}} \sum_{\substack{ad=n\\ d>0}} \sum_{j=0}^{d-1} \begin{cases} f\left(\dfrac{az+j}{d}\right) & \text{if } n>0, \\ f\left(\dfrac{a\bar{z}+j}{d}\right) & \text{if } n<0, \end{cases}$$

for non-zero $n \in \mathbb{Z}$. Up to scalar multiplication, there exists a countably infinite sequence of Maaß cusp forms which are also eigenfunctions for every Hecke operator T_n. We will denote this by $\{f_j\}_{j=1}^{\infty}$. Let $\lambda_j = \frac{1}{4} + r_j^2$ denote the Laplace eigenvalue and $a_j(n)$ denote the Hecke eigenvalues of f_j. That is

$$\Delta f_j = \left(\tfrac{1}{4} + r_j^2\right) f_j \quad \text{and} \quad T_n f_j = a_j(n) f_j.$$

The Hecke eigenvalues $a_j(n)$ are multiplicative, that is $a_j(mn) = a_j(m)a_j(n)$ for m and n coprime integers. In particular, $a_j(-n) = (-1)^{\omega_j} a_j(n)$ for some $\omega_j \in \{0,1\}$, called the parity of f_j. If $\omega_j = 0$ we say f_j is even and if $\omega_j = 1$ then we say f_j is odd. The r_j may be taken to be positive real numbers, and we may assume that the f_j are ordered such that $r_1 \leq r_2 \leq r_3 \leq \ldots$. Additionally, the f_j have a Fourier expansion of the form

$$f_j(x+iy) = \sum_{n=1}^{\infty} \frac{a_j(n)}{\sqrt{n}} W_{ir_j}(2\pi n y) \cos^{(\omega_j)}(2\pi n x),$$

where $W_{ir}(x) = \sqrt{x} e^{\frac{\pi}{2} r} K_{ir}(x)$, $K_{ir}(x)$ is the K-Bessel function, and we define $\cos^{(\omega)} = \cos$ if $\omega = 0$ and $\cos^{(\omega)} = -\sin$ if $\omega = 1$. We remark that the normalising factor $e^{\frac{\pi}{2} r}$ is non-standard; it is designed to compensate for the exponential decay of the K-Bessel function as $r \to \infty$ and is convenient for numerical purposes.

The Ramanujan conjecture predicts that the Hecke eigenvalues are bounded by the divisor function, $d(n)$. Unfortunately this is not yet known for Maaß forms, though we do have the estimate $|a_j(p)| \leq p^{7/64} + p^{-7/64}$ for primes p, due to Kim and Sarnak [19]. From the Hecke relations it follows that

$$|a_j(n)| \leq b(n) := \prod_{p^k \| n} \frac{\sinh((k+1)\theta \log p)}{\sinh(\theta \log p)}, \tag{2.1}$$

where $\theta = 7/64$.

Note that the constant function $f = 1$ is a solution to (1)–(3) above, as well as a Hecke eigenfunction with T_n eigenvalue $\sigma_{-1}(|n|)\sqrt{|n|}$. Although not a Maaß cusp form, it is part of the discrete spectrum of Δ on $L^2(\Gamma\backslash\mathbb{H})$, and will play a role in the proof of the Selberg trace formula (Proposition 2.1).

2.1. The Selberg trace formula. The Selberg trace formula is an expression for the weighted sum

$$\sum_{j=1}^{\infty} a_j(m) h(r_j),$$

where $n \in \mathbb{Z}\backslash\{0\}$ and h is a suitable test function (see Proposition 2.1 for more details). For us the key interest in this formula is that it involves the values $L(1,\chi)$ for quadratic Dirichlet characters χ, which are in turn related to quadratic class groups via Dirichlet's class number formula (see (2.2)).

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To state the formula precisely, we recall some notation from Section 1.1 in [1]. Let $\mathcal{D}$ denote the set of discriminants, that is

$$\mathcal{D} = \{D \in \mathbb{Z} : D \equiv 0 \text{ or } 1 \pmod 4\}.$$

Any non-zero $D \in \mathcal{D}$ can be uniquely expressed in the form $d\ell^2$, where d is a fundamental discriminant and $\ell > 0$. We define

$$\psi_D(n) = \left(\frac{d}{n/\gcd(n,\ell)}\right),$$

where $(-)$ denotes the Kronecker symbol. We see that ψ_D is periodic modulo D, and if D is a fundamental discriminant, then ψ_D is the usual quadratic character modulo D.

We set

$$L(s,\psi_D) = \sum_{n=1}^{\infty} \frac{\psi_D(n)}{n^s} \quad \text{for } \operatorname{Re}(s) > 1.$$

When we set $D = d\ell^2$, we can rewrite this as

$$L(s,\psi_D) = L(s,\psi_d) \prod_{p|\ell} \left[1 + (1-\psi_d(p)) \sum_{j=1}^{\operatorname{ord}_p(\ell)} p^{-js}\right].$$

Here we see that $L(s,\psi_D)$ has analytic continuation to $\mathbb{C}$, apart from a simple pole at $s = 1$ when D is square. When D is not a square, we have

$$L(1,\psi_D) = \frac{L(1,\psi_d)}{\ell} \prod_{p|\ell} \left[1 + (1-\psi_d(p)) \frac{(\ell,p^\infty)-1}{p-1}\right],$$

where (ℓ, p^∞) denotes the largest power of p that divides ℓ.

In turn, $L(1,\psi_d)$ is related to the class number and regulator of the quadratic field $\mathbb{Q}(\sqrt{d})$ by Dirichlet's class number formula:

$$L(1,\psi_d) = \frac{h_d}{\sqrt{|d|}} \cdot \begin{cases} \frac{2\pi}{w_d} & \text{if } d < 0, \\ 2R_d & \text{if } d > 0. \end{cases} \tag{2.2}$$

Here

$$w_d = \begin{cases} 2 & \text{if } d < -4, \\ 4 & \text{if } d = -4, \\ 6 & \text{if } d = -6 \end{cases}$$

is the number of roots of unity in the ring of integers of $\mathbb{Q}(\sqrt{d})$.

We can now state the Selberg trace formula for the modular group. The main reference for this proof is Section 2 of [1].

Proposition 2.1 (The Selberg trace formula for the modular group). *Let n be a non-zero integer and $f \in C^3(\mathbb{R})$ be even of compact support. Define*

$$h(r) = 2|n|^{-ir} \int_0^\infty f\left(v - \frac{n}{v}\right) v^{2ir} \frac{dv}{v} \quad \textit{for } r \in \mathbb{R},$$

and for $a \in \mathbb{N}$ with $a \mid n$ define

$$\Phi(a) = 2\sum_{m=1}^{\infty} \frac{\Lambda(m)}{m} f(am - \frac{n}{am}) + 2a\int_a^{\infty} \frac{f(v-\frac{n}{v}) - f(a-\frac{n}{a})}{v^2-a^2}\,dv$$
$$+ \big(\gamma + \log(4\pi)\big) f\Big(a - \frac{n}{a}\Big) - \frac{1}{2}\int_0^{\infty} f\Big(v - \frac{n}{v}\Big)\frac{dv}{v} - a^{-1}\int_{\mathbb{R}} f\Big(\sqrt{y^2 - \min(4n,0)}\Big)\,dy$$
$$+ \begin{cases} \displaystyle\sum_{\substack{m\in\mathbb{N}\\ m\mid(a-\frac{n}{a})}} \Lambda(m)\big(1-m^{-1}\big) f\Big(a-\frac{n}{a}\Big) + \int_{|a-\frac{n}{a}|}^{\infty} \frac{f(y)}{y+|a-\frac{n}{a}|}\,dy & \text{if } a \neq \frac{n}{a},\\ \displaystyle(\gamma - \log 2) f(0) + \frac{1}{2}\int_0^{\infty} \frac{f(y)+f(y^{-1})-f(0)}{y}\,dy + \frac{1}{3}\int_0^{\infty}\frac{f(0)-f(y)}{y^2}\,dy & \text{if } a = \frac{n}{a}, \end{cases}$$

where γ is the Euler–Mascheroni constant and $\Lambda(m)$ is the von Mangoldt function.

Then,

$$\sum_{j=1}^{\infty} a_j(n)h(r_j) = \sum_{\substack{a\in\mathbb{N}\\ a\mid n}} \Phi(a) + \sum_{\substack{t\in\mathbb{Z}\\ \sqrt{D}=\sqrt{t^2-4n}\notin\mathbb{Q}}} L(1,\psi_D)\cdot \begin{cases} f\big(\sqrt{D}\big) & \text{if } D>0,\\ \displaystyle\frac{\sqrt{|D|}}{\pi}\int_{\mathbb{R}} \frac{f(y)}{y^2+|D|}\,dy & \text{if } D<0. \end{cases}$$

Proof. Suppose first that f is smooth. In [1, Proposition 2.1][1] we find the following trace formula:

$$\sum_{j=0}^{\infty} a_j(n)h(r_j) = \sum_{\substack{a\in\mathbb{N}\\ a\mid n}} F(a) + \sum_{\substack{t\in\mathbb{Z}\\ \sqrt{D}=\sqrt{t^2-4n}\notin\mathbb{Z}}} W(D), \tag{2.3}$$

where

$$W(D) = \begin{cases} L(1,\psi_D) f\big(\sqrt{D}\big) & \text{if } 0<\sqrt{D}\notin\mathbb{Z},\\ \displaystyle L(1,\psi_D)\frac{\sqrt{|D|}}{\pi}\int_{\mathbb{R}}\frac{f(y)}{y^2+|D|}\,dy & \text{if } D<0,\\ \displaystyle\sum_{m\mid\sqrt{D}} \Lambda(m)(1-m^{-1}) f(\sqrt{D}) + \int_{\sqrt{D}}^{\infty}\frac{f(y)\,dy}{y+\sqrt{D}} & \text{if } 0<\sqrt{D}\in\mathbb{Z},\\ \displaystyle\frac{1}{2}(\gamma-\log 2) f(0) + \frac{1}{6}\int_0^{\infty}\frac{f(0)-f(y)}{y^2}\,dy & \\ \displaystyle+\frac{1}{4}\int_0^{\infty}\frac{f(y)+f(y^{-1})-f(0)}{y}\,dy & \text{if } D=0, \end{cases}$$

and

$$F(a) = 2\sum_{m=1}^{\infty}\frac{\Lambda(m)}{m} f\Big(am - \frac{n}{am}\Big) + 2a\int_a^{\infty}\frac{f(v-\frac{n}{v}) - f(a-\frac{n}{a})}{v^2-a^2}\,dv$$
$$+ \big(\gamma+\log(4\pi)\big) f\Big(a-\frac{n}{a}\Big) - \frac{1}{4}h(0).$$

Note that this includes an extra term $j=0$ corresponding to the constant eigenfunction 1, with $r_0 = i/2$ and $a_0(n) = \sigma_{-1}(|n|)\sqrt{|n|}$.

[1]There is a minor error in [1, Proposition 2.1]; the definition of $W(0)$ should be divided by 2.

To begin, we note that

$$h(0) = 2\int_0^\infty f\left(v - \frac{n}{v}\right)\frac{dv}{v}.$$

Thus, we see that $\Phi(a)$ and $F(a)$ only differ by the final line of $\Phi(a)$ and the term $-a^{-1}\int_{\mathbb{R}} f\left(\sqrt{y^2 - \min(4n, 0)}\right) dy$, which comes from the $j = 0$ term on the left-hand side of (2.3). Averaging the integral formulas for $h(i/2) = h(-i/2)$ and making the substitution $v \mapsto \frac{y+\sqrt{y^2+4|n|}}{2}$, we have

$$\begin{aligned}\sigma_{-1}(|n|)\sqrt{|n|}h\left(\frac{i}{2}\right) &= \sum_{\substack{a\in\mathbb{N}\\ a|n}} a^{-1}\int_0^\infty f\left(v - \frac{n}{v}\right)\left(1 + |n|v^{-2}\right) dv\\ &= \sum_{\substack{a\in\mathbb{N}\\ a|n}} a^{-1}\int_{\mathbb{R}} f\left(\sqrt{y^2 - \min(4n, 0)}\right) dy,\end{aligned}$$

as required.

As for the final line of $\Phi(a)$, we define the map

$$\begin{aligned}\left\{t \in \mathbb{Z} : \sqrt{t^2 - 4n} \in \mathbb{Z}\right\} &\to \{a \in \mathbb{N} : a \mid n\},\\ t &\mapsto a = \left|\frac{t + \sqrt{t^2 - 4n}}{2}\right|.\end{aligned}$$

Then for $t \in \mathbb{Z}$ with $\sqrt{t^2 - 4n} \in \mathbb{Z}$, we have that a is a positive divisor of n with $t^2 - 4n = (a - n/a)^2$. Furthermore, this map is a bijection unless n is a square, in which case the value $a = \sqrt{n}$ is assumed twice (from $t = \pm 2\sqrt{n}$). Hence the corresponding terms on the right-hand side of (2.3) contribute as the final line of $F(a)$. Note that the contribution from $a = n/a$ is doubled.

Finally, we remove the assumption from [**1**, Proposition 2.1] that the test function is smooth. Under our hypotheses on f, we can apply integration by parts three times to the definition of h to see that $h(r) \ll |r|^{-3}$. By the Weyl estimate $\#\{j : r_j \le r\} \ll r^2$, it follows that the left-hand side of (2.3) is absolutely convergent. The conclusion now follows by a straightforward approximation argument. □

We call the terms where $D > 0$ and $\sqrt{D} \notin \mathbb{Q}$ *hyperbolic* and the terms where $D < 0$ *elliptic*.

2.2. Specializing the test function. In order to apply the trace formula as a certification tool, it is necessary to choose a test function f that allows us to work out explicit expressions for the terms occurring in Proposition 2.1. For this we consider the test function

$$f(y) = \max\left(0, 1 - \frac{y^2}{X}\right)^k, \tag{2.4}$$

where $k \ge 4$ is an integer and X is a positive real number. We see this is an even, C^3 function that is supported on $[-\sqrt{X}, \sqrt{X}]$, so it satisfies our criteria in Proposition 2.1. The next proposition makes each term in the trace formula explicit for this test function.

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Proposition 2.2. *Let n, D be non-zero integers, a, X be positive real numbers and $k \geq 4$ be an integer. Assume that $D \geq -4n$ and $X > \max(D, (a - n/a)^2)$, and set*

$$b = \frac{\sqrt{X} + \sqrt{X + 4n}}{2a}, \quad A = \frac{n + ab\sqrt{X}}{|n|}, \quad x = \sqrt{X/|D|}.$$

Then, with f and h as defined in (2.4) and Proposition 2.1, we have

(i) $h(r) = 2 \cdot k! \left(\frac{A|n|}{X}\right)^k \sum_{j=0}^{k} (-1)^j \binom{k}{j} \operatorname{Re}\left(\frac{A^{ir-2j}}{\prod_{l=-j}^{k-j}(l + ir)}\right)$ *for $r \in \mathbb{R} \setminus \{0\}$;*

(ii) $h(0) = 2\int_0^\infty f\left(v - \frac{n}{v}\right)\frac{dv}{v} = 2\left(\frac{A|n|}{X}\right)^k \sum_{j=0}^{k} \binom{k}{j}^2 A^{-2j}\left(\log A + \sum_{l=1}^{j}\frac{1}{l} - \sum_{l=1}^{k-j}\frac{1}{l}\right)$;

(iii) $\int_{\mathbb{R}} f\left(\sqrt{y^2 - \min(4n, 0)}\right) dy = 2\sqrt{X}\left(1 + \frac{\min(4n, 0)}{X}\right)^{k+\frac{1}{2}} \prod_{j=1}^{k} \frac{2j}{2j+1}$;

(iv) $\int_{\sqrt{|D|}}^\infty \frac{f(y)\, dy}{y + \sqrt{|D|}} = (1 - x^{-2})^k \log\left(\frac{x+1}{2}\right) - \sum_{j=0}^{k}\binom{k}{j}(-x^2)^{-j} \sum_{l=1}^{2j}(-1)^{l-1}\frac{x^l - 1}{l}$;

(v) $\sqrt{|D|}\int_{\mathbb{R}} \frac{f(y)\, dy}{y^2 + |D|} = 2(1 + x^{-2})^k \arctan x - 2\sum_{j=0}^{k}\binom{k}{j} x^{-2j} \sum_{l=1}^{j} \frac{(-1)^{l-1}}{2l-1} x^{2l-1}$;

(vi) $$2a\int_a^\infty \frac{f(v - \frac{n}{v}) - f(a - \frac{n}{a})}{v^2 - a^2}\, dv = f\left(a - \frac{n}{a}\right)\log\left(\frac{b-1}{b+1}\right)$$
$$+ 2\sum_{m=-k}^{k}\left(-\frac{a^2}{n}\right)^m \sum_{j=|m|}^{k}\binom{k}{j}\binom{2j}{j+|m|}\left(\frac{n}{X}\right)^j \sum_{l=1}^{|m|}\frac{b^{(2l-1)\operatorname{sgn} m} - 1}{2l-1};$$

(vii) $\int_0^\infty \frac{f(0) - f(y)}{y^2}\, dy = \frac{2k+1}{\sqrt{X}}\prod_{j=1}^{k}\frac{2j}{2j+1}$;

(viii) $\int_0^\infty \frac{f(y) + f(y^{-1}) - f(0)}{y}\, dy = \log X - \sum_{j=1}^{k}\frac{1}{j}$.

Proof. Using the definition of h and making the change of variables $v \mapsto \sqrt{|n|u}$, we have

$$\begin{aligned} h(r) &= 2|n|^{-ir}\int_0^\infty \max\left(0, 1 - \frac{v^2 - 2n + (n/v)^2}{X}\right)^k v^{2ir}\frac{dv}{v} \\ &= \int_0^\infty \max\left(0, 1 + \frac{2n}{X} - \frac{|n|}{X}\left(u + u^{-1}\right)\right)^k u^{ir}\frac{du}{u} \\ &= \left(\frac{|n|T}{X}\right)^k \int_0^\infty \max\left(0, 1 - \frac{u + u^{-1}}{T}\right)^k u^{ir}\frac{du}{u}, \end{aligned}$$

where $T = \frac{X+2n}{|n|} \geq 2$.

Now let $F : [2, \infty) \to \mathbb{C}$ be a k-times differentiable function of compact support and let $s \in \mathbb{C}$. Applying integration by parts inductively, we derive

$$\int_0^\infty F\left(u + u^{-1}\right)u^s\frac{du}{u} = \int_0^\infty F^{(k)}\left(u + u^{-1}\right)\sum_{j=0}^{k}(-1)^j\binom{k}{j}\frac{(s + 2j - k)u^{s+2j-k}}{\prod_{l=0}^{k}(s + j - l)}\frac{du}{u}.$$

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We can apply this to our specific test function $F(t) = \max\left(0, 1 - \frac{t}{T}\right)^k$, noting that

$$F^{(k)}(t) = \begin{cases} \dfrac{(-1)^k k!}{T^k} & \text{if } t < T, \\ 0 & \text{if } t > T. \end{cases}$$

Thus, using the above formula we obtain

$$\begin{aligned} \int_0^\infty \max\left(0, 1 - \frac{u+u^{-1}}{T}\right)^k u^{ir} \frac{du}{u} &= \frac{(-1)^k k!}{T^k} \int_{1/A}^{A} \sum_{j=0}^{k} (-1)^j \binom{k}{j} \frac{(ir+2j-k)u^{ir+2j-k}}{\prod_{l=0}^{k}(ir+j-l)} \frac{du}{u} \\ &= \frac{(-1)^k k!}{T^k} \sum_{j=0}^{k} (-1)^j \binom{k}{j} \frac{A^{ir+2j-k} - A^{-ir-2j+k}}{\prod_{l=0}^{k}(ir+j-l)}, \end{aligned}$$

where $A = \frac{T+\sqrt{T^2-4}}{2} = \frac{X+2n+\sqrt{X^2+4nX}}{2|n|}$, so that $A + A^{-1} = T$. Replacing (j, l) by $(k - j, k - l)$ in the above sum, we see that it becomes

$$\frac{2k!}{T^k} \sum_{j=0}^{k} (-1)^{k-j} \binom{k}{j} \operatorname{Re}\left(\frac{A^{ir+2j-k}}{\prod_{l=0}^{k}(ir+j-l)}\right).$$

Now multiplying by $\left(\frac{|n|T}{X}\right)^k$ and replacing (j, l) by $(k - j, k - j - l)$ in the above sum yields (i).

To evaluate $h(0)$, we write

$$h(r) = \frac{A^{ir}H(r) - A^{-ir}H(-r)}{2ir},$$

where

$$H(r) = 2 \cdot k! \left(\frac{A|n|}{X}\right)^k \sum_{j=0}^{k} (-1)^j \binom{k}{j} A^{-2j} \prod_{\substack{-j \le l \le k-j \\ l \neq 0}} (l + ir)^{-1}.$$

By l'Hôpital's rule, we have

$$h(0) = (\log A)H(0) - iH'(0).$$

Hence, a straightforward evaluation of $H(0)$ and $H'(0)$ gives (ii).

Next, making the substitution $y \mapsto u\sqrt{X + \min(4n, 0)}$, we obtain

$$\begin{aligned} \int_{\mathbb{R}} f\left(\sqrt{y^2 - \min(4n, 0)}\right) dy &= \int_{\mathbb{R}} \max\left(0, 1 - \frac{y^2 - \min(4n, 0)}{X}\right)^k dy \\ &= \sqrt{X}\left(1 + \frac{\min(4n, 0)}{X}\right)^{k+\frac{1}{2}} \int_{-1}^{1} (1 - u^2)^k du, \end{aligned}$$

which yields (iii).

For the next term, making the substitution $y \mapsto u\sqrt{X}$, we have

$$\int_{\sqrt{|D|}}^{\sqrt{X}} \frac{f(y)dy}{y + \sqrt{|D|}} = \int_{x^{-1}}^{1} \frac{(1-u^2)^k}{u + x^{-1}} du.$$

Writing $(1-u^2)^k = (1-x^{-2})^k + (1-u^2)^k - (1-x^{-2})^k$ and applying the binomial theorem to the last two terms, we get

$$\int_{x^{-1}}^{1} \frac{(1-u^2)^k}{u+x^{-1}} du = \int_{x^{-1}}^{1} \frac{(1-x^{-2})^k}{u+x^{-1}} du + \sum_{j=0}^{k} \binom{k}{j} (-1)^j \int_{x^{-1}}^{1} \frac{u^{2j}-x^{-2j}}{u+x^{-1}}\, du.$$

Expanding the right-most integrand into a geometric series, we obtain

$$\frac{u^{2j}-x^{-2j}}{u+x^{-1}} = -x^{1-2j} \sum_{l=1}^{2j} (-xu)^{l-1}.$$

Integrating each term of this sum over $[x^{-1}, 1]$ yields (iv).

Similarly,

$$\begin{aligned}
\sqrt{|D|} \int_{\mathbb{R}} \frac{f(y)\, dy}{y^2+|D|} &= x^{-1} \int_{-1}^{1} \frac{(1-u^2)^k}{u^2+x^{-2}}\, du \\
&= x^{-1} \int_{-1}^{1} \frac{(1+x^{-2})^k}{u^2+x^{-2}}\, du + x^{-1} \sum_{j=0}^{k} \binom{k}{j} \int_{-1}^{1} \frac{(-u^2)^j - (x^{-2})^j}{u^2+x^{-2}}\, du \\
&= 2\big(1+x^{-2}\big)^k \arctan x - 2 \sum_{j=0}^{k} \binom{k}{j} x^{-2j} \sum_{l=1}^{j} \frac{(-1)^{l-1}}{2l-1} x^{2l-1}.
\end{aligned}$$

For (vi), we begin by noting that $(v-n/v)^2 \le X$ for $a \le v \le ab$, hence $f(n-n/v) = (1-(v-n/v)^2/X)^k$ in this region. Applying the binomial theorem twice, we find that

$$\frac{f(v-\frac{n}{v}) - f(a-\frac{n}{a})}{v^2-a^2} = \sum_{j=0}^{k} \binom{k}{j} \left(\frac{n}{X}\right)^j \sum_{m=-j}^{j} \binom{2j}{j+|m|} (-n)^{-m} \frac{v^{2m}-a^{2m}}{v^2-a^2}.$$

Now, expanding the right-most fraction into a geometric series, we find that

$$a \int_{a}^{ab} \frac{v^{2m}-a^{2m}}{v^2-a^2} dv = a^{2m} \sum_{l=1}^{|m|} \frac{b^{(2l-1)\operatorname{sgn} m} - 1}{2l-1}.$$

Inserting this into the above equation and rearranging the sum over the values of m to go between $m = -k$ to k yields the first part of (vi). The second part arises from the contribution of the integral over $v > ab$, where $f(v-n/v) = 0$. That is,

$$2a \int_{ab}^{\infty} \frac{f\left(v-\frac{n}{v}\right) - f\left(a-\frac{n}{a}\right)}{v^2-a^2} dv = -f\left(a-\frac{n}{a}\right) \int_{b}^{\infty} \frac{2\, du}{u^2-1} = -f\left(a-\frac{n}{a}\right) \log \frac{b+1}{b-1}.$$

Turning to (vii), we use integration by parts and the substitution $y \mapsto u\sqrt{X}$ to obtain

$$\int_{\mathbb{R}} \frac{f(0)-f(y)}{y^2}\, dy = -\int_{\mathbb{R}} \frac{f'(y)}{y}\, dy = \frac{2k}{\sqrt{X}} \int_{-1}^{1} \big(1-u^2\big)^{k-1}\, du = \frac{4k+2}{\sqrt{X}} \prod_{j=1}^{k} \frac{2j}{2j+1}.$$

Finally, for (viii), we have

$$\int_0^\infty \frac{f(y)+f(y^{-1})-f(0)}{y}\,dy = 2\int_0^1 \frac{f(y)+f(y^{-1})-f(0)}{y}\,dy$$
$$= 2\int_0^1 \frac{f(y)-f(0)}{y}\,dy + 2\int_1^\infty \frac{f(y)}{y}\,dy$$
$$= 2\int_0^{\sqrt{X}} \frac{f(y)-f(0)}{y}\,dy + 2f(0)\int_1^{\sqrt{X}} \frac{1}{y}\,dy.$$

Now using the substitution $y \mapsto u\sqrt{X}$ and noting that $f(0) = 1$, this becomes

$$\log X - 2\int_0^1 \frac{1-(1-u^2)^k}{u}\,du = \log X - \int_0^1 \frac{1-v^k}{1-v}\,dv = \log X - \sum_{j=1}^k \frac{1}{j}.$$

□

2.3. Idea of the algorithm. As noted before, we see the class number for real quadratic fields appearing in the hyperbolic terms in the Selberg trace formula in Proposition 2.1. The main idea of our algorithm is to compute the spectral side of the trace formula with known Maaß form data, bound its tail and see if the two sides of the trace formula match with our class group data. For this section we shall assume that we are using the test function f defined by (2.4).

To begin, suppose we have rigorously computed values for r_j and $\lambda_j(n)$ for $j \le J$ and $|n| \le M$, so that we may compute the spectral side of the trace formula to high accuracy. There will be some error arising from the terms with $j > J$, for which we have no data. More details on how to explicitly estimate the tail of the spectral sum will be given in Section 2.3.1, but suppose for now that we can bound the tail by some positive real number E_n. By the explicit form of the trace formula we derived in Proposition 2.1, we have

$$\begin{aligned}(2.5)\qquad &\sum_{\substack{t\in\mathbb{Z}\\ D=t^2-4n<0}} L(1,\psi_D)\frac{\sqrt{|D|}}{\pi}\int_{\mathbb{R}}\frac{f(y)\,dy}{y^2+|D|} + \sum_{\substack{t\in\mathbb{Z}\\ \sqrt{D}=\sqrt{t^2-4n}\notin\mathbb{Q}\\ 0<D\le X}} L(1,\psi_D)\left(1-\frac{D}{X}\right)^k \\ &= \sum_{j=1}^\infty \lambda_j(n)h(r_j) - \sum_{\substack{a\in\mathbb{N}\\ a\mid n}}\Phi(a) \\ &\le \sum_{j=1}^J \lambda_j(n)h(r_j) + E_n - \sum_{\substack{a\in\mathbb{N}\\ a\mid n}}\Phi(a).\end{aligned}$$

Now, suppose we have a list of class numbers computed using our conditional algorithm. *A priori* we do not know that the class numbers are correct, but we know that each computed value is a factor of the true value (being the order of some subgroup of the class group). Hence our data can be used to compute a rigorous lower bound for the left-hand side of (2.5), since the terms are non-negative. (In order to compute $L(1,\psi_D)$ for $D > 0$, we also need the corresponding regulators. Although the fastest algorithms for that also rely on GRH, they can be independently verified using the method of [10]. Hence we may assume that the regulators are known unconditionally.)

Moreover, any incorrect value must be off by at least a factor of 2. Hence, in order to certify a given class number, we just need to show that it is not at least twice as large as we think it is. To this end, we double the corresponding term in the hyperbolic sum and then compute the full hyperbolic sum. If the sum exceeds the right-hand side of (2.5) then we get a contradiction, and hence our purported value of the class number must have been correct. Heuristically we expect the truncation error to be much smaller than our rigorous estimate E_n, so we expect to be able to certify all d for which $L(1,\psi_d)(1-d/X)^k$ exceeds E_n. Note that considering all $n \in \mathbb{Z}\setminus\{0\}$ with $|n| \le \frac{1}{2}\sqrt{X-1}$ suffices to cover all non-square discriminants $d \le X$.

In our case, we have the first 2184 Laplace eigenvalues with $r \in (0, 177.75]$ computed by Andreas Strömbergsson using Hejhal's algorithm [**13**] and certified using the program from [**5**]. The proof of their completeness is given in Corollary 1.2 in [**2**]. In Section 3 we use a rigorous version of Hejhal's "Phase 2" algorithm to compute all of the needed Hecke eigenvalues, $a_j(n)$. The next few sections discuss how to explicitly bound the tail of the spectral sum, and estimate the efficiency of the algorithm with our given data.

2.3.1. *Bounding the tail of the spectral sum.* In order to apply the above algorithm we require an explicit bound on the tail of the spectral sum. To begin, using Proposition 2.2 (i), we have that

$$|h(r)| \le \frac{2\cdot k!}{|r|^{k+1}},$$

which becomes sharp in the limit as $X \to \infty$. Using this estimate, we can bound the tail of the spectral sum without needing specific estimates of the terms of the trace formula. Namely, we need to find an explicit bound for the sum

$$\sum_{j:r_j>R} r_j^{-k-1},$$

for some positive real R.

The main idea here is to use the fact that the eigenvalue counting function, $N(t) = \#\{j : r_j \le t\}$, is majorized by its Weyl asymptotic. More precisely, let

$$M(t) = \frac{t^2}{12} - \frac{2t}{\pi}\log\frac{t}{e\sqrt{\frac{\pi}{2}}} - \frac{131}{144} \quad \text{and} \quad S(t) = N(t) - M(t).$$

Then, from [**12**, Ch. 10, Thm. 2.29] we have

$$S(t) = O\left(\frac{t}{\log t}\right) \qquad \text{for } t > 1.$$

In order to apply this numerically, we require an explicit constant for the big-O. Currently this has not been worked out, however we can remedy this by using an integrated version derived in [**2**, Theorem 1.1]. Explicitly, define

$$S_1(t) = \frac{1}{t}\int_0^t S(u)\,du \quad \text{and} \quad E(t) = \left(1 + \frac{6.59125}{\log t}\right)\left(\frac{\pi}{12\log t}\right)^2.$$

Then,

$$S_1(t) \le E(t) \quad \text{for all } t > 1. \tag{2.6}$$

Consider

$$\sum_{j:r_j>R} r_j^{-k-1} = \int_R^\infty t^{-k-1}\,dN(t) = \int_R^\infty t^{-k-1}M'(t)\,dt + \int_R^\infty t^{-k-1}\,dS(t).$$

Applying integration by parts to the last integral twice, the above becomes

$$\sum_{j:r_j>R} r_j^{-k-1} = \int_R^\infty t^{-k-1}M'(t)\,dt - \frac{S(R)+(k+1)S_1(R)}{R^{k+1}} + (k+1)(k+2)\int_R^\infty t^{-k-2}S_1(t)\,dt.$$

Using the bound (2.6) and our explicit form of $M(t)$, we obtain

$$\sum_{j:r_j>R} r_j^{-k-1} \le \frac{1}{6(k-1)R^{k-1}} - \frac{2\log(R\sqrt{2/\pi})+2/k}{\pi k R^k} - \frac{S(R)+(k+1)S_1(R)}{R^{k+1}} + \frac{(k+2)E(R)}{R^{k+1}}.$$

For given values of R and k, we can easily check that the non-principal terms contribute a negative amount. Thus, using our data with $R \le 177$ and $k \le 15$, we find that

$$\sum_{j:r_j>R} r_j^{-k-1} \le \frac{R^{1-k}}{6(k-1)}.$$

Using this and the bound on the Hecke eigenvalues (2.1) due to Kim and Sarnak, we can bound the tail by

$$\left|\sum_{j:r_j>R} \lambda_j(n)h(r_j)\right| \le b(n)\sum_{j:r_j>R}|h(r_j)| \le 2b(n)k!\sum_{j:r_j>R} r_j^{-k-1} \le \frac{b(n)k!}{3(k-1)}R^{1-k}. \tag{2.7}$$

2.3.2. *Efficiency.* We can use our explicit bound of the spectral tail (2.7) to get an idea of how efficient this algorithm will be. We will be able to certify a given d provided that the corresponding hyperbolic term on the right-hand side of (2.5) exceeds the amount that we overestimate the tail by. More explicitly, we should get

$$L(1,\psi_d)\left(1-\frac{d}{X}\right)^k > \frac{b(n)k!}{3(k-1)}R^{1-k} - \sum_{j:r_j>R}\lambda_j(n)h(r_j).$$

We do not know the sum over j in advance, but we expect it to be much smaller than our estimate (2.7). Thus, we should succeed in certifying d as long as

$$\frac{d}{X} \lessapprox 1 - \frac{1}{R}\left(\frac{b(n)k!R}{3(k-1)L(1,\psi_d)}\right)^{1/k}.$$

For instance, if $X = 10^{11}$ then the worst case value of $b(n)$ is $164.397\ldots$, attained at $n = 151200$. If we assume that $L(1,\chi_d)$ has roughly the same minimum value as among the negative discriminants up to 10^{11} (viz., 0.17448, as computed in [16]), then the optimal k is 11, for which the above is about 94%. However, already with $k = 6$ we get 92%, and that may allow us to get by with significantly lower floating point precision. (Note that the total sum over d has size roughly $\sqrt{X}$, but we are trying to detect variations of size $L(1,\chi_d)(1-d/X)^k$, which can be less than 10^{-7} even with $k = 6$. Hence it is also essential that we work with interval arithmetic in order to control for cancellation; we made use of the `Arb` library [18] for this purpose.) This analysis is also highly pessimistic in assuming that the worst case for $b(n)$ occurs simultaneously with the worst case for $L(1,\chi_d)$.

For these reasons, we ran our verification with $k = 6$. We made two runs on a machine with 64 cores (2.5 GHz AMD Opteron processors), with the following results:

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X	certified up to	running time
1.1×10^{10}	10 378 129 942	5 hours
1.1×10^{11}	103 455 923 536	57 hours

In both runs, the efficiency was better than 94%, and about 1.3% of the computation time was spent on the right-hand side of (2.5).

2.3.3. *Theoretical complexity.* The computation of the Maaß forms is possible in polynomial time, [**25**, §1.3.4]. Since we can take k arbitrarily large in the above analysis, the eigenvalue cutoff R can grow slowly as a function of X, and the time to compute the spectral side is therefore dominated by the computation of the Hecke eigenvalues, which is $O(X^{\frac{1}{2}+\varepsilon})$ for each form (see Section 3).

Thus, the slowest part of the computation of the right-hand side of (2.5) is the sum over m appearing in Φ, which has roughly $\frac{\sqrt{X}}{a \log X}$ non-zero terms. Summing over $a \mid n$ and $|n| \le \sqrt{X}$ gives $O(\frac{X}{\log X})$ terms in total. However, this is still swamped by the roughly X terms appearing on the left-hand side of (2.5) in the hyperbolic sum. This motivates our choice of test function, which makes the hyperbolic terms simple to compute.

As described in Section 4, the algorithm that we employ to compute the class group and regulator work in time $O(X^{\frac{5}{4}+\varepsilon})$ overall. Asymptotically one could turn to an index calculus based algorithm with heuristic complexity $O(X^{1+\varepsilon})$. Unfortunately, the correctness of the index calculus approach depends on GRH in several ways, and there is currently no known method of verifying its output in subexponential time.

3. Rigorous computation of the Hecke eigenvalues

In order to compute the truncated sum on the spectral side of the trace formula, we require a large list of Hecke eigenvalues for each of the Laplace eigenvalues. As noted before, we have approximations of the Laplace eigenvalues of the first 2184 Maaß forms of level 1, as well as a rigorously verified list of the first several Hecke eigenvalues for each form. All this data has been computed and verified to better than 300 bits of precision, which allows us to compute a given Maaß form $f(z)$ for any z in the fundamental domain to approximately this accuracy. In turn, we can compute many more Hecke eigenvalues using the "Phase 2" part of Hejhal's algorithm [**13**]. In this section we explain how to carry out the Phase 2 algorithm rigorously.

Let f be a Maaß cusp form on $\mathrm{PSL}(2,\mathbb{Z})$ with Laplace eigenvalue $\lambda = \frac{1}{4} + r^2$ and Hecke eigenvalues a_m. Let $\omega = 0$ if f is even and $\omega = 1$ if f is odd. Its Fourier expansion is of the form

$$f(x+iy) = \sum_{m=1}^{\infty} \frac{a_m}{\sqrt{m}} W_{ir}(2\pi m y) \cos^{(\omega)}(2\pi m x)$$

where $W_{ir}(x) = \sqrt{x} e^{\frac{\pi}{2} r} K_{ir}(x)$ and $K_{ir}(x)$ is the K-Bessel function. In addition, $\cos^{(\omega)} = \cos$ if $\omega = 0$ and $\cos^{(\omega)} = -\sin$ if $\omega = 1$.

Fix $N \in \mathbb{N}, Y > 0$ and define the $2N$ points

$$z_j = x_j + iY = \frac{j - \frac{1}{2}}{2N} + iY,$$

where $1-N \le j \le N$. Now if we consider the discrete Fourier transform of f, for some integer k, on these points we get

$$\begin{aligned}\sum_{j=1}^{N} f(z_j)\cos^{(\omega)}(2\pi k x_j) &= \sum_{j=1}^{N}\sum_{m=1}^{\infty} \frac{a_m}{\sqrt{m}} W_{ir}(2\pi m Y)\cos^{(\omega)}(2\pi m x_j)\cos^{(\omega)}(2\pi k x_j)\\ &= \sum_{m=1}^{\infty} \frac{a_m}{\sqrt{m}} W_{ir}(2\pi m Y)\sum_{j=1}^{N}\cos^{(\omega)}(2\pi m x_j)\cos^{(\omega)}(2\pi k x_j).\end{aligned}$$

Here we can use the trigonometric identity $\cos^{(\omega)}(x)\cos^{(\omega)}(y) = \frac{1}{2}\cos(x-y)+(-1)^{\omega}\cos(x+y)$, to obtain

$$\begin{aligned}&\sum_{j=1}^{N} f(z_j)\cos^{(\omega)}(2\pi k x_j)\\ &= \frac{1}{2}\sum_{m=1}^{\infty}\frac{a_m}{\sqrt{m}}W_{ir}(2\pi m Y)\left[\sum_{j=1}^{N}\cos(2\pi(m-k)x_j) + (-1)^{\omega}\sum_{j=1}^{N}\cos(2\pi(m+k)x_j)\right].\end{aligned} \tag{3.1}$$

Here our goal is extract the k-th term of the series on the right-hand side and then get an expression for the rest of the sum which we will bound later. We have

$$\begin{aligned}\sum_{j=1}^{N}\cos(2\pi(m\pm k)x_j) &= \frac{1}{2}\sum_{j=1}^{N}\left(e^{2\pi(m\pm k)x_j} + e^{-2\pi(m\pm k)x_j}\right)\\ &= \frac{1}{2}e^{-\frac{m\pm k}{2N}\pi i}\sum_{j=1}^{N} e^{2\pi i\frac{(m\pm k)j}{2N}} + \frac{1}{2}e^{\frac{m\pm k}{2N}\pi i}\sum_{j=1}^{N} e^{-2\pi i\frac{(m\pm k)j}{2N}}.\end{aligned}$$

Now if $2N \mid (m\pm k)$, then $\sum_{j=1}^{N} e^{\pm 2\pi i\frac{(m\pm k)j}{2N}} = N$. Otherwise, using the fact that this sum is a geometric series, we get 0. Thus we can simplify the above sum to

$$\sum_{j=1}^{N}\cos(2\pi(m\pm k)x_j) = \begin{cases}(-1)^{\frac{(m\pm k)}{2N}}N & \text{if } 2N \mid (m\pm k),\\ 0 & \text{otherwise.}\end{cases} \tag{3.2}$$

Hence combining the results of (3.1) and (3.2), we have

$$\begin{aligned}\frac{2}{N}\sum_{j=1}^{N} f(z_j)\cos^{(\omega)}(2\pi k x_j) &= \sum_{\substack{m\ge 1\\ m\equiv k(2N)}}\frac{a_m}{\sqrt{m}}W_{ir}(2\pi m Y)(-1)^{\frac{(m-k)}{2N}}\\ &\quad + (-1)^{\omega}\sum_{\substack{m\ge 1\\ m\equiv -k(2N)}}\frac{a_m}{\sqrt{m}}W_{ir}(2\pi m Y)(-1)^{\frac{(m+k)}{2N}}\\ &= \frac{a_k}{\sqrt{k}}W_{ir}(2\pi k Y) + \mathcal{E}_0,\end{aligned}$$

where

$$\mathcal{E}_0 = \sum_{j=1}^{\infty}(-1)^{j}\left[\frac{a_{2jN+k}}{\sqrt{2jN+k}}W_{ir}(2\pi(2jN+k)Y) + (-1)^{\omega}\frac{a_{2jN-k}}{\sqrt{2jN-k}}W_{ir}(2\pi(2jN-k)Y)\right]. \tag{3.3}$$

In order for the above truncation to be valid we require $k \le N$. Let z_j^* be the pullback of z_j into the fundamental domain defined by $\left\{z = x+iy \in \mathbb{H} \mid |z| \ge 1 \text{ and } |x| \le \frac{1}{2}\right\}$. Then by the modularity of f, we have $f(z_j) = f(z_j^*)$. Thus

$$\begin{aligned}\frac{a_k}{\sqrt{k}} W_{ir}(2\pi kY) &= \frac{2}{N}\sum_{j=1}^{N} f(z_j^*)\cos^{(\omega)}(2\pi k x_j) - \mathcal{E}_0 \\ &= \frac{2}{N}\sum_{j=1}^{N}\sum_{m=1}^{\infty} \frac{a_m}{\sqrt{m}} W_{ir}(2\pi m y_j^*)\cos^{(\omega)}(2\pi m x_j^*)\cos^{(\omega)}(2\pi k x_j) - \mathcal{E}_0 \\ &= \frac{2}{N}\sum_{j=1}^{N}\left(\sum_{m=1}^{L} \frac{a_m}{\sqrt{m}} W_{ir}(2\pi m y_j^*)\cos^{(\omega)}(2\pi m x_j^*) + \mathcal{E}_j\right)\cos^{(\omega)}(2\pi k x_j) - \mathcal{E}_0,\end{aligned}$$

where $L \in \mathbb{N}$ is a truncation parameter, and

$$\mathcal{E}_j = \sum_{m=L+1}^{\infty} \frac{a_m}{\sqrt{m}} W_{ir}(2\pi m y_j^*)\cos^{(\omega)}(2\pi m x_j^*). \tag{3.4}$$

Here we can consider the total error given by

$$\mathcal{E} = \frac{2}{N}\sum_{j=1}^{N} \mathcal{E}_j \cos^{(\omega)}(2\pi k x_j) - \mathcal{E}_0, \tag{3.5}$$

and our main computation formula becomes

$$\frac{a_k}{\sqrt{k}} W_{ir}(2\pi kY) = \frac{2}{N}\sum_{j=1}^{N}\sum_{m=1}^{L} \frac{a_m}{\sqrt{m}} W_{ir}(2\pi m y_j^*)\cos^{(\omega)}(2\pi m x_j^*)\cos^{(\omega)}(2\pi k x_j) + \mathcal{E}. \tag{3.6}$$

Computationally, we can see this is just a discrete cosine/sine transformation with respect to the Hecke eigenvalues. Thus, once we have values of Y and N, discussed in Subsection 3.2, we can apply a standard computational library on Fast Fourier Transforms to compute these sums.

Our goal now is to bound the total error $\mathcal{E}$ explicitly so that it can aid us in our computations.

3.1. Bounding the error.

To begin, we have

$$|\mathcal{E}| \le \left|\frac{2}{N}\sum_{j=1}^{N} \mathcal{E}_j \cos^{(\omega)}(2\pi k x_j)\right| + |\mathcal{E}_0| \le 2 \max_{1\le j\le N}\{|\mathcal{E}_j|\} + |\mathcal{E}_0|.$$

We now want to bound the individual parts appearing in the above bound. For this we require the following two lemmas. The first is bound on the Fourier coefficients from Kim–Sarnak [19].

LEMMA 3.1. *Let f be a Maaß cusp form of level 1 with Hecke eigenvalues a_m. Then for all non-zero $m \in \mathbb{Z}$ we have*

$$\left|\frac{a_m}{\sqrt{m}}\right| \le \eta := 1.758.$$

PROOF. Using (2.1), we have $\left|\frac{a_m}{\sqrt{m}}\right| \le \frac{b(m)}{\sqrt{m}}$, and this is maximized at $m = 12$. □

The second lemma we require is a bound on the K-Bessel function due to Booker, Strömbergsson and Then [4, Prop. 1].

Lemma 3.2. *For all $y > r > 0$ we have*

$$|W_{ir}(y)| = e^{\frac{\pi}{2}r}\sqrt{y}|K_{ir}(y)| \leq \sqrt{\frac{\pi}{2}}\frac{\sqrt{y}}{\sqrt[4]{y^2-r^2}}e^{-ru(y/r)},$$

where $u(t) = \sqrt{t^2-1} - \arctan(\sqrt{t^2-1})$.

We can now directly apply the above lemmas to bound the sums appearing in $\mathcal{E}$.

Proposition 3.3. *Let b_m be an increasing arithmetic sequence for $m_0 \leq m < \infty$ with $b_{m_0} > r$ and arithmetic difference d. Then*

$$\sum_{m=m_0}^{\infty}\left|\frac{a_m}{\sqrt{m}}W_{ir}(b_m)\right| < \mathcal{B}_{r,b_{m_0},d} := \eta\sqrt{\frac{\pi}{2}}\frac{\sqrt{b_{m_0}}}{\sqrt[4]{b_{m_0}^2-r^2}}e^{-ru(b_{m_0}/r)}\left(1+\frac{b_{m_0}}{d\sqrt{b_{m_0}^2-r^2}}\right),$$

where $u(t) = \sqrt{t^2-1} - \arctan(\sqrt{t^2-1})$.

Proof. We begin by noting that the function $\frac{\sqrt{y}}{\sqrt[4]{y^2-r^2}}$ is decreasing for $y > r$. Hence by applying both of the above lemmas we get

$$\sum_{m=m_0}^{\infty}\left|\frac{a_m}{\sqrt{m}}W_{ir}(b_m)\right| < \eta\sqrt{\frac{\pi}{2}}\frac{\sqrt{b_{m_0}}}{\sqrt[4]{b_{m_0}^2-r^2}}\sum_{m=m_0}^{\infty}e^{-ru(b_m/r)}.$$

The goal here is to majorize the exponential sum by a geometric series. For this, we note that the function $e^{-ru(y/r)}$ is decreasing for $y > r$ and $u'(t) = \sqrt{1-t^{-2}}$ is increasing for $t > 1$. Hence for all $t_2 > t_1 > 1$, we have

$$u(t_2) \geq u(t_1) + (t_2-t_1)u'(t) = u(t_1) + (t_2-t_1)\sqrt{1-t_1^{-2}}.$$

Thus, for all $m \geq m_0$ we obtain

$$ru(b_m/r) \geq ru(b_{m_0}/r) + \sqrt{b_{m_0}^2-r^2}\frac{b_m-b_{m_0}}{b_{m_0}}.$$

We can now bound the exponential sum by

$$\begin{aligned}\sum_{m=m_0}^{\infty}\exp(-ru(b_m/r)) &\leq \exp(-ru(b_{m_0}/r))\exp(\sqrt{b_{m_0}^2-r^2})\sum_{m=m_0}^{\infty}\exp\left(-\frac{\sqrt{b_{m_0}^2-r^2}}{b_{m_0}}b_m\right)\\ &\leq \exp(-ru(b_{m_0}/r))\left(1-\exp\left(-\frac{\sqrt{b_{m_0}^2-r^2}}{b_{m_0}}d\right)\right)^{-1}.\end{aligned}$$

To get the final result we use the fact that $(1-e^{-x})^{-1} < 1 + x^{-1}$ for $x > 0$. ☐

Using Proposition 3.3 we can compute bounds for the errors $\mathcal{E}_0$ and $\mathcal{E}_j$.

Proposition 3.4. *Let $L, M \in \mathbb{N}$ with $0 < k \leq M < N$, $2\pi Y(2N-M) > r$, and $\sqrt{3}\pi(L+1) > r$. Then we have*

$$\begin{aligned}|\mathcal{E}_0| &\leq 2\mathcal{B}_{r,2\pi Y(2N-M),4\pi YN},\\ |\mathcal{E}_j| &\leq \mathcal{B}_{r,\sqrt{3}\pi(L+1),\sqrt{3}\pi}\end{aligned}$$

for all $1 \leq j \leq N$.

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PROOF. From Lemma 3.2, we see that $|W_{ir}(y)|$ is decreasing for $y > r$. Now using the definition of $\mathcal{E}_0$ from (3.3), we have

$$\begin{aligned}|\mathcal{E}_0| &\le \left|\sum_{j=1}^{\infty} \frac{a_{2jN+k}}{\sqrt{2jN+k}} W_{ir}(2\pi(2jN+k)Y) + (-1)^{\omega} \sum_{j=1}^{\infty} \frac{a_{2jN-k}}{\sqrt{2jN-k}} W_{ir}(2\pi(2jN-k)Y)\right| \\ &\le \sum_{j=1}^{\infty} \left|\frac{a_{2jN+1}}{\sqrt{2jN+1}} W_{ir}(2\pi(2jN+1)Y)\right| + \sum_{j=1}^{\infty} \left|\frac{a_{2jN-M}}{\sqrt{2jN-M}} W_{ir}(2\pi(2jN-M)Y)\right|.\end{aligned}$$

Thus applying Proposition 3.3 we obtain the result.

For $\mathcal{E}_j$, we note that since all the z_j^* are in the fundamental domain, we have $y_j^* > \sqrt{3}/2$ for all j. Hence from the definition of $\mathcal{E}_j$ from (3.4) we get

$$|\mathcal{E}_j| \le \sum_{m=L+1}^{\infty} \left|\frac{a_m}{\sqrt{m}} W_{ir}(2\pi m y_j^*)\right| < \mathcal{B}_{r,2\pi(L+1)y_j^*,2\pi y_j^*} \le \mathcal{B}_{r,\sqrt{3}\pi(L+1),\sqrt{3}\pi},$$

by Proposition 3.3. □

In practice we choose L to be the number of initial Fourier coefficients that we know. We ensure this is sufficiently large that the error is dominated by our estimate for $|\mathcal{E}_0|$, i.e. that $\mathcal{B}_{r,\sqrt{3}\pi(L+1),\sqrt{3}\pi} \le \mathcal{B}_{r,2\pi Y(2N-M),4\pi YN}$.

3.2. Choosing Y and N. For our code, we let M be the largest indexed Fourier coefficient we wish to compute. We will only need to consider the Fourier coefficients a_p for $p \le M$ prime since the others can be computed using the Hecke relations from this data. To help control the error we have to carefully choose the parameters Y and N. To begin we note that the $W_{ir}(y)$ decays exponentially for $y > r$ from the K-Bessel function.

We start by choosing $Y = r/M$. Then we compute $W_{ir}(2\pi pY)$ for all primes $p \le M$. The aim of this is to see if we are near any of the zeros of the K-Bessel function in its oscillatory region, which would cause our error bound to blow up. If we are too close to a zero, we can change Y slightly so that we move away from this zero. However, we have to make sure we do not make any other values of $W_{ir}(2\pi pY)$ close to a different zero. This is essentially a min-max problem of minimizing the value of Y whilst maximizing the distance of the values of $W_{ir}(2\pi pY)$ away from zero.

Once we have a value for Y, we can work on finding N. To do this we first fix a precision of B bits, and then we bound our error $|\mathcal{E}|$ to be roughly 2^{-B}, that is

$$|\mathcal{E}| \le 4\mathcal{B}_{r,2\pi Y(2N-M),4\pi YN} = 2^{-B}.$$

Note that in practice we will want to choose B larger than our desired output precision due to rounding errors and the fact we will be dividing by W_{ir}. Now, we know all the constants r, Y, M and B, hence we can rearrange the above to become

$$Q(N) := \frac{1}{\eta}\sqrt{\frac{2}{\pi}}\mathcal{B}_{r,2\pi Y(2N-M),4\pi YN} = \frac{1}{\eta}\sqrt{\frac{2}{\pi}}2^{-B-2}.$$

Hence to find N, we just need to find the root of

$$Q(N) - \frac{1}{\eta}\sqrt{\frac{2}{\pi}}2^{-B-2},$$

which we find by bisection.

4. Tabulation algorithm

Although the index-calculus algorithm of Buchmann [6] remains the best method in terms of asymptotic complexity for computing class groups and regulators of real quadratic fields, it is not necessarily the best choice for our application. The fields under our consideration all have small discriminants, almost certainly too small for the asymptotic superiority of Buchmann's algorithm to be in effect.

Following [15], we instead used a combination of an unconditionally correct algorithm for computing the regulator and a generic group structure algorithm for computing the elementary divisors of the class group (and hence also the class number), modified for class groups of real quadratic fields. We used Lenstra's improvement [20] of Shanks' baby-step giant-step algorithm [24], as described in [17, Sec 10.2], which compute an unconditionally correct approximation of the regulator in time $O(\Delta^{1/5+\varepsilon})$ under the extended Riemann hypothesis. For computing the class group, we used an algorithm due to Buchmann and Schmidt [8], modified as follows as was required to apply it to class groups of real quadratic fields.

Termination. The input of the group structure algorithms in [7] and [8] is a set of group elements $g_1, \ldots, g_l$, and the output is the subgroup generated by them. If $g_1, \ldots, g_l$ is known to generate the entire class group, then the output is the structure of the entire class group. One way to construct such a generating set is to take as the g_i all prime ideals of norm less than a suitable bound, but unfortunately the best known bounds, even assuming Riemann hypotheses [11], are of the form $c \log^2 \Delta$ for some constant $c < 4$. This is much too large for these baby-step giant-step based algorithms, especially when considering that in many cases the class group is cyclic and only one or two small-norm prime ideals already generates all of it.

Instead, we follow the procedure of [17, Sec 10.4] and iteratively enlarge the subgroup one prime ideal at a time in order of increasing norm. To determine when the the entire class group is generated, we first compute the regulator (using Lenstra's algorithm as stated above), and then use an approximation of the L-function $L(1, \chi_\Delta)$ and the analytic class number formula to compute a bound h^* such that $h^* < h_\Delta R_\Delta < 2h^*$. Then, as soon as the order of the subgroup generated is such that the product of that and the regulator exceeds h^*, we know that we have the entire class group.

This approach has the advantage that the class group is often obtained very quickly with few prime ideals required. It also allows us to collect data on the largest prime ideals required to generate each class group, that we can use for empirical evidence in support of the GRH-dependent bounds such as [11]. However, to compute the bound h^* sufficiently quickly we have to assume the extended Riemann hypothesis to bound the error in our approximation of $L(1, \chi_\Delta)$. Thus, the output of the algorithm is only correct under the GRH. However, we do know that because the algorithm always computes a subgroup of the class group that if any of our class groups are wrong, their orders are all less than the true class numbers, an important fact for our verification method to work correctly.

Equivalence testing. Both of the generic class group algorithms assume that testing the equality of group elements can be done efficiently. However, in the case of class groups of real quadratic fields this is not the case, as there is no known computationally-efficient way to identify unique canonical representatives of ideal equivalence classes. The best known approach is to use reduced ideals to represent class group elements. Computing a reduced ideal equivalent to a given ideal is fast and it can be shown that every equivalence class contains finitely many reduced ideals, but the number of reduced ideals in each class is $O(R_\Delta)$. In our modification of [8], we employ an equivalence test based on Shanks' infrastructure by storing roughly $\sqrt{R_\Delta}$ equivalent reduced ideals along with each baby step (corresponding to infrastructure baby steps) and checking whether each giant step and roughly $\sqrt{R_\Delta}$ equivalent ideals corresponding to infrastructure giant steps.

Complexity. The group structure algorithm of [8] has complexity $O(\sqrt{G_l}\Delta^\varepsilon)$, and our modified version thus has cost $O(\Delta^{1/4+\varepsilon})$ because we require $\sqrt{h_\Delta}$ group operations and $\sqrt{R_\Delta}$ infrastructure operations to test equivalence for each of those for a total of $O((h_\Delta R_\Delta)^{1/2+\varepsilon})$ bit operations. Note that the correctness and running time of both methods are dependent on the GRH, due to the method used to determine termination described above.

5. Numerical results

An initial data set was tabulated on machines at the University of Bristol, across 256 cores (AMD Opteron processors), using `PARI` [28]. The verification data set tabulations were performed on the University of Calgary's ARC supercomputer. Specifically, the computation was distributed across 3 nodes of the cpu2022 partition, which have Intel Xeon Gold 5320 processors and overall provided 156 cores to distribute across. All ARC nodes run an up-to-date version of Rocky Linux 8, and the tabulation program used ANTL [14], which was built and run using `g++ 8.5.0`, `GMP 6.2.1`, `MPFR 4.1.0`, `Open MPI 4.1.1`, `MPICH 3.4.3` and `NTL 11.5.1`.

5.1. Tabulation. Tabulations were completed in two parts. First, regulators and class groups were computed for all $\Delta \leq 1.1 \times 10^{11}$ using `PARI`, and took a real time of 1 week. Regulator and class group computations were carried out for all $\Delta \leq 1.1 \times 10^{11}$. For the latter, total CPU time spent was 1379 days and real time spent was 649 hours. The size of the tabulated data set was approximately 345GB after compressing using `Gzip 1.9`. Using a simple verification program, the results of the `PARI` were confirmed against the verification data set for all $\Delta \leq 1.1 \times 10^{11}$, and took 31 hours.

5.2. The Cohen–Lenstra heuristics. Let $\mathrm{Cl}_\Delta^{\mathrm{odd}}$ denote largest odd-order subgroup of Cl_Δ, and let h_Δ^{odd} be its order. For primes $p \geq 3$, Cohen and Lenstra [9] conjectured that the probability that a given abelian p-group G occurs as the p-Sylow subgroup of $\mathrm{Cl}_\Delta^{\mathrm{odd}}$ is proportional to $(\#G \cdot \#\operatorname{Aut} G)^{-1}$. Assuming this, it was shown in [17, §7.8, Conj. 7.8] that

$$\Pr(h_\Delta^{\mathrm{odd}} = h) = \rho(h) := \frac{C}{h^2} \prod_{p^r \| h} \frac{1}{\eta_r(p)},$$

where

$$\eta_r(p) = \prod_{i=1}^{r} (1 - p^{-i}), \quad \eta_\infty(p) = \lim_{r \to \infty} \eta_r(p), \quad \text{and} \quad C = \frac{1}{2\eta_\infty(2) \prod_{r=2}^{\infty} \zeta(r)},$$

and that

$$\Pr(\mathrm{Cl}_\Delta^{\mathrm{odd}} \text{ has } p\text{-rank } r) = \frac{\eta_\infty(p)}{p^{r(r+1)} \eta_r(p) \eta_{r+1}(p)},$$

from which we derive

$$\begin{aligned} &\Pr(\mathrm{Cl}_\Delta^{\mathrm{odd}} \text{ has rank } r) = \delta(r) \\ &\quad := C \left(\prod_{p \geq 3} \sum_{k=0}^{r} \frac{1 - p^{-1}}{p^{k(k+1)} \eta_k(p) \eta_{k+1}(p)} - \prod_{p \geq 3} \sum_{k=0}^{r-1} \frac{1 - p^{-1}}{p^{k(k+1)} \eta_k(p) \eta_{k+1}(p)} \right). \end{aligned}$$

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TABLE 5.1. Comparison of actual and expected frequencies of odd class number h over $\Delta \le 10^{11}$

h	$\rho(h)$	$\#\{\Delta \le 10^{11} : h_\Delta^{\mathrm{odd}} = h\}$	$\frac{1}{\rho(h)} \cdot \frac{\#\{\Delta \le 10^{11} : h_\Delta^{\mathrm{odd}} = h\}}{\#\{\Delta \le 10^{11}\}}$
1	0.7544582	23077857263	1.0063263
3	0.1257430	3720399446	0.9733839
5	0.0377229	1148230625	1.0013881
7	0.0179633	548292832	1.0041652
9	0.0157179	458686616	0.9600650
11	0.0068587	209526693	1.0050215
13	0.0048363	147794143	1.0053682
15	0.0062872	185045389	0.9682842

TABLE 5.2. Comparison of actual and expected frequencies of odd rank r over $\Delta \le 10^{11}$

r	$\delta(r)$	$\#\{\Delta \le 10^{11} : \mathrm{Cl}_\Delta^{\mathrm{odd}} \text{ has rank } r\}$	$\frac{1}{\delta(r)} \cdot \frac{\#\{\Delta \le 10^{11} : \mathrm{Cl}_\Delta^{\mathrm{odd}} \text{ has rank } r\}}{\#\{\Delta \le 10^{11}\}}$
0	0.7544582	23077857263	1.0063263
1	0.2431724	7256625197	0.9817455
2	2.366145×10^{-3}	61821672	0.8595633
3	3.282595×10^{-6}	51015	0.5112804
4	5.078341×10^{-10}	1	0.0647823

Values of $\rho(h)$ and $\delta(r)$ are shown in Tables 5.1 and 5.2, and compared with the observed frequencies for $\Delta \le 10^{11}$. Plainly, some of the observed counts are much closer to their expected values than others, e.g. we see that class numbers divisible by 3 and ranks ≥ 2 are significantly rarer than expected. This is explained in Figures 5.1 and 5.2, which show the evolution of each ratio over $\Delta \le X$, on a log scale. Trend fitting of the curves supports the conclusion that, in each case, the ratio tends to 1 as $X \to \infty$, albeit at a slow rate of convergence. For instance, the best fit to $\frac{1}{\rho(3)} \cdot \frac{\#\{\Delta \le X : h_\Delta^{\mathrm{odd}} = 3\}}{\#\{\Delta \le X\}}$ over $X \in [10^{10}, 10^{11}]$ by a curve of the form $a - bX^{-c}$ is shown in Figure 5.3, and indicates a limiting value of $a \approx 1.003327$ and error rate $\approx O(X^{-0.139222})$.

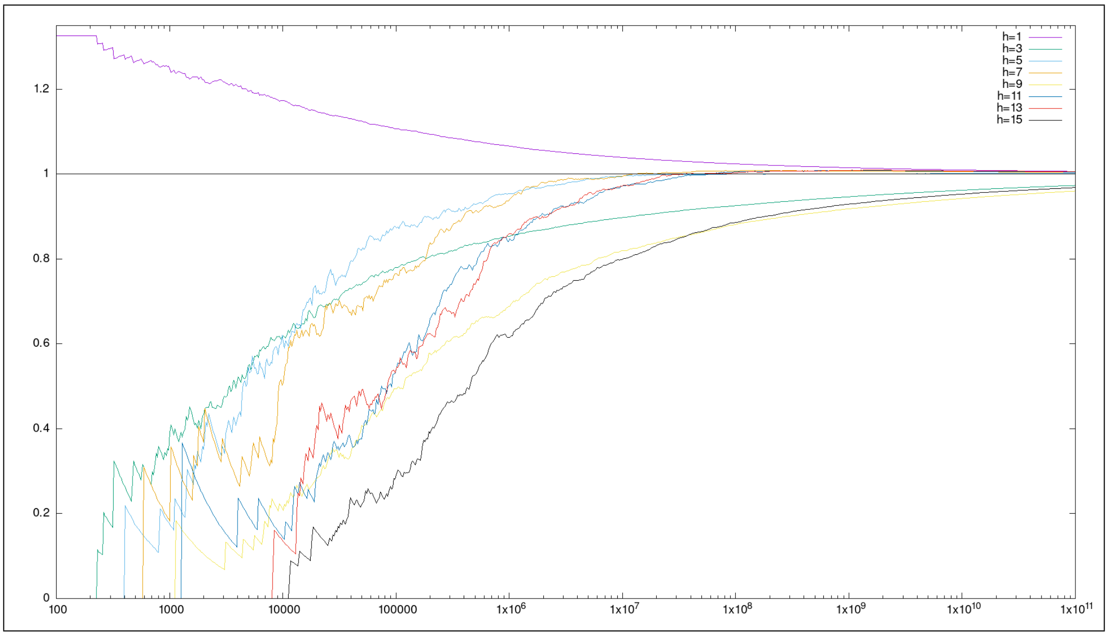

FIGURE 5.1. Ratio of actual and expected frequencies of odd class number h over $\Delta \leq X$

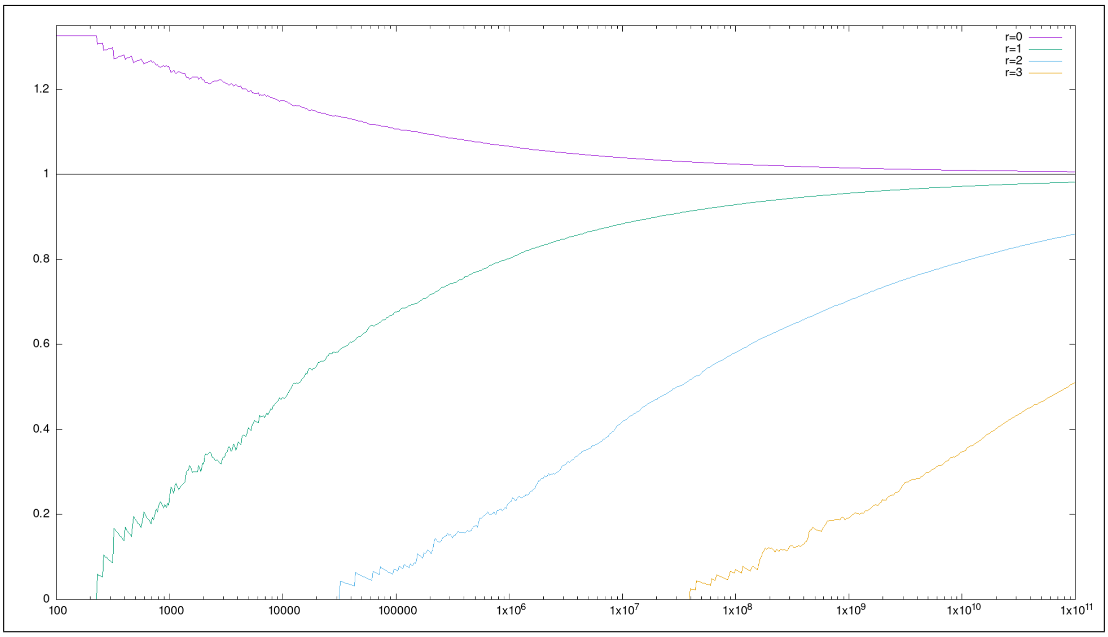

FIGURE 5.2. Ratio of actual and expected frequencies of odd rank r over $\Delta \leq X$

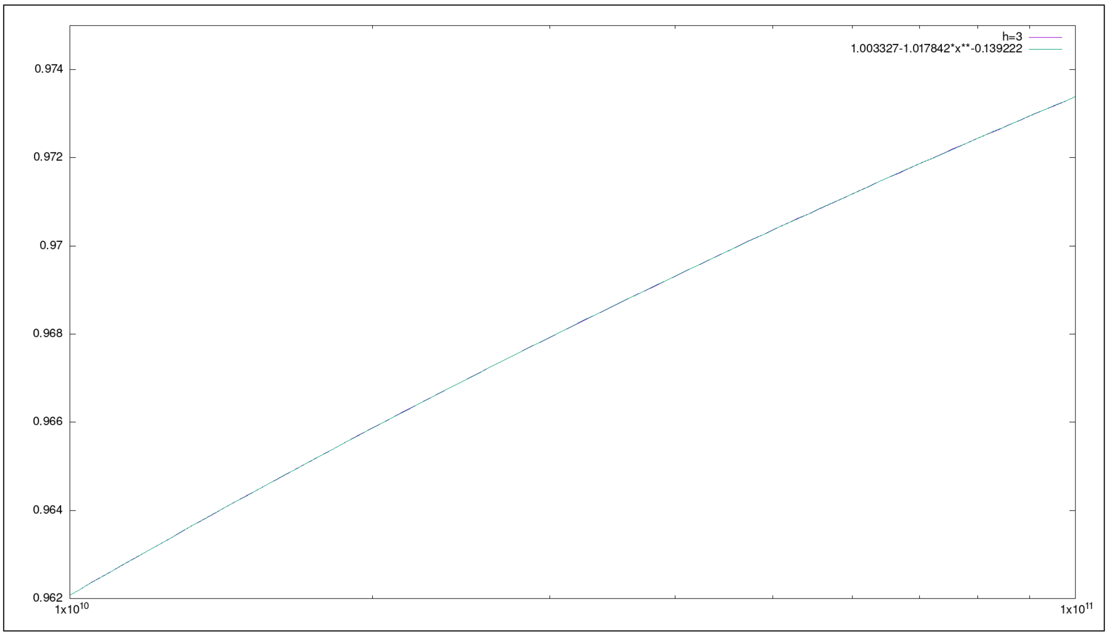

FIGURE 5.3. $\frac{1}{\rho(3)} \cdot \frac{\#\{\Delta \le X : h_\Delta^{\mathrm{odd}} = 3\}}{\#\{\Delta \le X\}}$ compared with $1.003327 - 1.017842X^{-0.139222}$

6. Conclusions

We have shown that unconditionally certifying the table of class groups and regulators using Maaß forms works very well in practice. Indeed, the bottleneck in terms of extending our table further is producing the class groups and regulators. One possible improvement is to compute the class numbers using Shanks's algorithm of complexity $O(\Delta^{1/5+\varepsilon})$ [23] (see also [17, Sec 10.3], and then compute group structures only for possibly non-cyclic examples by applying Sutherland's algorithm for computing the structure of finite abelian p-groups [26] for each prime p dividing the class number with multiplicity greater than one. Since the Cohen-Lenstra heuristics imply that class numbers of real quadratic fields are small, and that fields with class groups of large rank are very rare, we expect that this method would work very well in practice.

Another intriguing option is to find an analogue of the method used by Mosunov and Jacobson [21] for unconditionally tabulating class groups of imaginary quadratic fields using class number generating functions. This method reduced the problem of computing class numbers to multiplication of large degree polynomials, and was shown to work very well in practice. It is not immediately clear whether similar formulas exist that could be used for real quadratic fields.

Finally, there has been very little work done on unconditional tabulation, nor table verification, for fields of degree greater than 2. Trace formulas of higher rank automorphic forms could possibly yield a verification algorithm analogous to ours, but there is much work to be done before that will become a reality.

7. Code availability

Source code for the algorithms presented here is available at

https://github.com/BBDJSH-LuCaNT-2023/

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